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The Square Element Graph over a Ring

Raibatak Sen Gupta
Department of Mathematics, Jadavpur University, Kolkata-700032, India
Email: raibatak2010@gmail.com

M. K. Sen
Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata-700019, India
Email: senmk6@yahoo.com

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Abstract. In this paper, we generalize the square element graph $\mathbb{S}q(R)$ by defining it over any ring R with unity. For a ring R with 1, $\mathbb{S}q(R)$ is defined as follows: it is a simple undirected graph where the vertex set is $R-\{0\}$, and two vertices are adjacent if and only if $a \neq b$ and $a+b=x^2$ for some $x \in R-\{0\}$. Using the results of $\mathbb{S}q(R)$ found earlier for finite commutative rings, here we first obtain some results regarding direct products of rings. Then we study $\mathbb{S}q(R)$ for infinite rings R. In particular, we obtain some results regarding connectedness, cycles and other properties of $\mathbb{S}q(\mathbb{Z})$. We also look at $\mathbb{S}q(\mathbb{Z}[x]), \mathbb{S}q(\mathbb{Z}_2[x])$, and $\mathbb{S}q(F)$, where F is any infinite field.

Keywords: Square element; Direct product; Finite field; Infinite graph; Complete graph.

1. Introduction

Graphs have been associated with algebraic structures in several ways (e.g.- in [1, 2, 4, 5]). In particular, interesting graph-theoretical structures of the set of zero-divisors of a ring have been revealed by studying the zero-divisor graph. This motivated us to define another interesting graph over a ring R using the set $S=\{x^2 \mid x \in R-\{0\}\}$. Like the set of zero-divisors, the set S is not closed under addition in general. Also, this is multiplicatively closed for a commutative ring without zero-divisors. We call an element of the set S a *square element* of the ring R.

For a finite field F, each non-zero element of F can be expressed as a sum of 2 squares. So if F is a field of characteristic 2, then $a = x^2 + y^2 = (x+y)^2$ for all $a \in F - \{0\}$. Since a is non-zero, we have that $x + y \neq 0$. Hence, every non-zero element in a finite field of characteristic 2 is a square element. Suppose F is of odd characteristic p, i.e., $|F| = p^n$ for some odd prime p, and $n \in \mathbb{N}$. We recall that $(F - \{0\}, .)$ is a cyclic group. So there exists some $g \in F - \{0\}$, such that $F - \{0\} = \{g, g^2, g^3, \ldots, g^{p^n-1}(=1)\}$. It is easy to see that the squares of non-zero elements in F are precisely the even powers of g, and the non-squares in F are precisely the odd powers of g (note that since F is a field, it has no zero-divisors, and hence the square of a non-zero element cannot be 0). So there are $\frac{p^n-1}{2}$ squares (of non-zero elements) and $\frac{p^n-1}{2}$ non-squares in F.

Walter Stangl [7] denoted the number of elements in the set $\{x^2 \mid x \in \mathbb{Z}_n\}$ by $S(n)$. He showed that $S(n)$ is a multiplicative function, and gave the following formulae for $S(n)$:

(i) The number of squares in $\mathbb{Z}_{2^n}$ is given by

$$S(2^n) = \begin{cases} \frac{2^{n-1}+4}{3} & \text{if } n \text{ is even;} \\ \frac{2^{n-1}+5}{3} & \text{if } n(\geq 3) \text{ is odd.} \end{cases}$$

(ii) For an odd prime p, the number of squares in $\mathbb{Z}_{p^n}$ is given by

$$S(p^n) = \begin{cases} \frac{p+1}{2} & \text{if } n = 1; \\ \frac{p^2-p+2}{2} & \text{if } n = 2; \\ \frac{p^{n+1}+p+2}{2(p+1)} & \text{if } n(>2) \text{ is even;} \\ \frac{p^{n+1}+2p+1}{2(p+1)} & \text{if } n(>2) \text{ is odd.} \end{cases}$$

The units in $\mathbb{Z}_n$ that are square elements are commonly called quadratic residues mod n. The theory of quadratic residues forms an important part of number theory. Clearly, it is worthwhile to study the interplay between the algebraic properties of squares in $\mathbb{Z}_n$ and the structure of any interesting graph associated with it.

In view of all these, we defined a graph called the *Square element graph* (denoted by $\mathbb{S}q(R)$) over a finite commutative ring R in our earlier paper [6]. Here, we generalize the graph $\mathbb{S}q(R)$ by defining it over any ring with identity (i.e., we are no longer restricting R to be a finite commutative ring). In this paper, we will mainly study commutative rings (both finite and infinite). We mention some of the earlier results obtained in our previous paper [6], and use them to get some new results, specially on direct product of rings. We also look at some properties of $\mathbb{S}q(R)$ defined over infinite rings like $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{Z}_2[x]$.

In this paper, $Char(R)$ denotes the characteristic of the ring R. For a positive integer m, by $\mathbb{Z}_m$ we mean the ring of integers modulo m. $\mathbb{F}_n$ denotes the finite field containing exactly n elements. $Q(D)$ denotes the quotient field of any integral domain D. Note that for all rings discussed in this paper, the unity 1 is distinct from the zero-element 0, unless mentioned otherwise. Also, $v \leftrightarrow w$ denotes that the vertices v and w are adjacent. For algebraic terminology, we refer to any standard book of Ring theory. For number-theoretic terms and graph-theoretical terminology, one may see [3] and [8], respectively.

2. The Graph $\mathbb{S}q(R)$ and Some Examples

For a ring R with 1, the square element graph over R is defined as follows:

Definition 2.1. *Let R be a ring with 1. Then the Square element graph over R is the simple undirected graph $G = (V, E)$, where $V = R - \{0\}$ and $ab \in E$ if and only if $a \neq b$ and $a + b = x^2$ for some $x \in R - \{0\}$. We denote this graph by $\mathbb{S}q(R)$.*

Let us look at some examples of the graph $\mathbb{S}q(R)$.

Example 2.2. The graphs $\mathbb{S}q(\frac{\mathbb{Z}_2[x]}{<x^2>})$, $\mathbb{S}q(\mathbb{Z}_4)$, and $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_2)$ are shown in figures 1, 2, 3, respectively. These three figures show that all non-trivial (i.e., not edgeless) simple graphs of 3 vertices (upto isomorphism) can be realized as $\mathbb{S}q(R)$. In the figures, a vertex of the form $g(x)+ < f(x) >$ is labelled simply as $g(x)$.

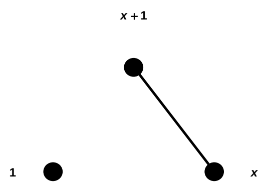

Figure 1: $\mathbb{S}q(\frac{\mathbb{Z}_2[x]}{<x^2>})$

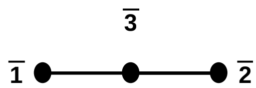

Figure 2: $\mathbb{S}q(\mathbb{Z}_4)$

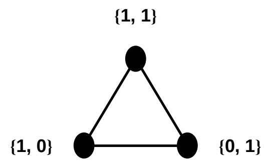

Figure 3: $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_2)$

Example 2.3. The graphs $\mathbb{S}q(\mathbb{F}_4), \mathbb{S}q(\mathbb{F}_8)$, and $\mathbb{S}q(\mathbb{F}_9)$ are shown in figures 4, 5, 6, respectively. The first two graphs are complete but the third is not. Note that $\mathbb{F}_4 \cong \frac{\mathbb{Z}_2[x]}{<x^2+1>}, \mathbb{F}_9 \cong \frac{\mathbb{Z}_3[x]}{<x^2+1>}$ and $\mathbb{F}_8 \cong \frac{\mathbb{Z}_2[x]}{<x^3+x+1>}$.

Remark 2.4. $\mathbb{S}q(R_1) \cong \mathbb{S}q(R_2)$ does not imply that $R_1 \cong R_2$. Figures 3 and 4 show that $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{S}q(\mathbb{F}_4)$, although $\mathbb{Z}_2 \times \mathbb{Z}_2 \ncong \mathbb{F}_4$.

Example 2.5. The graphs $\mathbb{S}q(\frac{\mathbb{Z}_2[x]}{<x^2,xy,y^2>})$, $\mathbb{S}q(\mathbb{Z}_3)$, and $\mathbb{S}q(\mathbb{Z}_5)$ are shown in Fig-

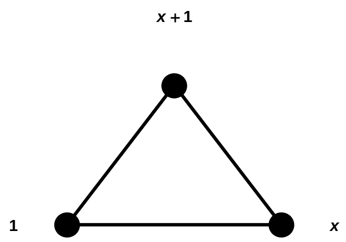

Figure 4: $\mathbb{S}q(\mathbb{F}_4)$

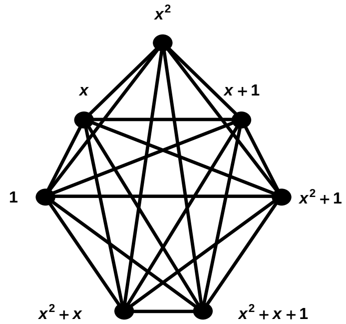

Figure 5: $\mathbb{S}q(\mathbb{F}_8)$

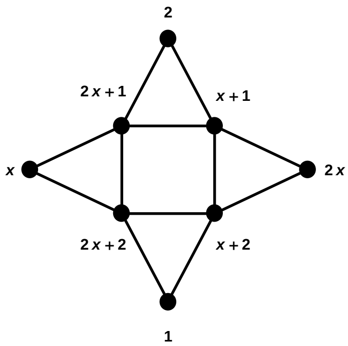

Figure 6: $\mathbb{S}q(\mathbb{F}_9)$

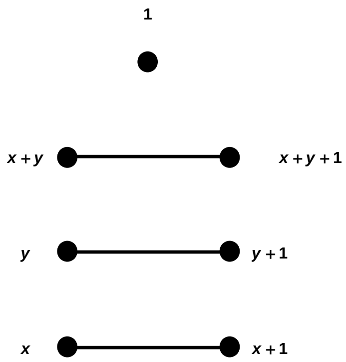

Figure 7: $\mathbb{S}q(\frac{\mathbb{Z}_2[x,y]}{<x^2,xy,y^2>})$

Figure 8: $\mathbb{S}q(\mathbb{Z}_3)$

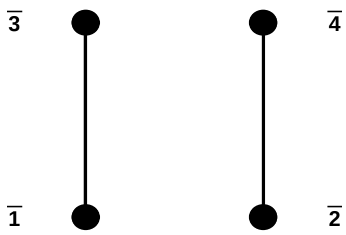

Figure 9: $Sq(\mathbb{Z}_5)$

ures 7, 8, 9, respectively. All these graphs are disconnected. It is interesting to note that $\mathbb{Z}_3, \mathbb{Z}_5$ are fields.

From the definition of $\mathbb{S}q(R)$, it is clear that we can take R to be non-commutative also. We now give such an example.

Example 2.6. Consider the ring $M_2(\mathbb{Z}_2)$, i.e., the ring of 2×2 matrices where the entries of the matrices are elements of $\mathbb{Z}_2$. So the entries of any matrix are either $\bar{0}$ or $\bar{1}$. Considering any matrix (b_{ij}), we denote the positions $11, 12, 21$ and 22 by positions $1, 2, 3, 4$ respectively. We denote the zero-matrix $\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$ in $M_2(\mathbb{Z}_2)$ as A_0. If a non-zero matrix A has entry $\bar{1}$ in positions $i_1, i_2, \ldots, i_s$ (where $s \leq 4$, $i_t \in \{1, 2, 3, 4\}$ for $t = 1, 2, \ldots, s$ and $i_1 < i_2 < \cdots < i_s$), then we denote A by $A_{i_1, i_2, \ldots, i_s}$. For example, the matrix $\begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix}$ is denoted by $A_{1,2,3,4}$; whereas the identity matrix I_2 in $M_2(\mathbb{Z}_2)$ is denoted by $A_{1,4}$. It is easy to see that the square elements in $M_2(\mathbb{Z}_2)$ are $A_0, A_1, A_4, A_{1,2}, A_{1,4}, A_{3,4}, A_{1,3}, A_{2,4}, A_{2,3,4}$, and $A_{1,2,3}$ (in this notation). Now in $\mathbb{S}q(M_2(\mathbb{Z}_2))$, we have a path $A_4 \leftrightarrow A_1 \leftrightarrow A_{1,4} \leftrightarrow A_{2,3,4} \leftrightarrow A_{1,2,3}$. Also, $A_{1,2}, A_{1,3}, A_{2,4}, A_{3,4}$ are each adjacent to $A_{1,4}$. So we have a connected subgraph containing the square vertices (note that A_0 is not a vertex). Again, noting that all nonsquares (i.e., $A_2, A_3, A_{2,3}, A_{1,3,4}, A_{1,2,4}, A_{1,2,3,4}$) are adjacent to A_1, it is easy to see that $\mathbb{S}q(M_2(\mathbb{Z}_2))$ is connected. However, it is not a complete graph, since $A_{1,3} \not\leftrightarrow A_{2,4}$.

3. Some Properties of $\mathbb{S}q(R)$ When R is Finite

In this section, we state some results obtained in our earlier paper [6], where the graph $\mathbb{S}q(R)$ is taken over finite commutative rings (with 1) only. These results are stated since they are used in the subsequent sections of this paper. We start with some properties of $\mathbb{S}q(\mathbb{Z}_n)$. We note that in Stangl's [7] expression $S(n)$, $\bar{0}$ was always taken as a square. However, according to our definition of square element, $\bar{0}$ is a square element if and only if $\mathbb{Z}_n$ has some $x \neq \bar{0}$ such that $x^2 = \bar{0}$ (or equivalently, n is not square-free). So we defined a new expression.

Definition 3.1. [6] *For $n \in \mathbb{N}, n > 1$, the number $S^*(n)$ denotes the number of elements (in $\mathbb{Z}_n$) of the form $\overline{x}^2$, where $\overline{x} \in \mathbb{Z}_n - \{\overline{0}\}$. So $S^*(n) = S(n) - 1$ or $S(n)$, according as n is square-free or not.*

Theorem 3.2. [6] *For any vertex $\overline{v}$ in $\mathbb{S}q(\mathbb{Z}_n)$, $deg(\overline{v})$ is $S^*(n), S^*(n) - 1$, or $S^*(n) - 2$ according as none of, exactly one of, or both of v and $2v$ are square elements.*

In general, for a finite commutative ring, we have the analogous result as given below.

Proposition 3.3. [6] *For a finite commutative ring R with 1 ($\neq 0$), the degree of any vertex in $\mathbb{S}q(R)$ is one of the following three numbers: $S^*(R)$, $S^*(R)-1$, $S^*(R)-2$, where $S^*(R)$ is the number of elements in the set $\{y^2 : y \in R-\{0\}\}$*

Next, we identify some classes of graphs for which $\mathbb{S}q(R)$ is connected.

Theorem 3.4. [6] *$\mathbb{S}q(\mathbb{Z}_n)$ is connected for all $n \in \mathbb{N} - \{1, 3, 5\}$.*

Theorem 3.5. [6] *$\mathbb{S}q(F)$ is connected for any finite field F, unless F is isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_5$.*

Theorem 3.6. [6] *If R is a finite boolean ring (i.e., R is a direct product of finitely many copies of $\mathbb{Z}_2$), then $\mathbb{S}q(R)$ is complete.*

Now we state the following result about completeness of $\mathbb{S}q(F)$ for a finite field F.

Theorem 3.7. [6] *For a finite field F, $\mathbb{S}q(F)$ is a complete graph if and only if $Char(F) = 2$.*

Proposition 3.8. [6] *If a ring R with 1 ($\neq 0$) has characteristic $n > 2$, then there exists a non-square element in R.*

The following can be obtained using the last proposition and noting that 1 is a square element in a ring with unity.

Proposition 3.9. [6] *If the characteristic of R is greater than 2, then $\mathbb{S}q(R)$ is regular if and only if for each $v \in R - \{0\}$, exactly one of v and $2v$ is a square, and in that case, it is $S^*(R) - 1$ regular.*

In particular, for $\mathbb{Z}_n$, we have the following result.

Theorem 3.10. [6] *$\mathbb{S}q(\mathbb{Z}_n)$ is regular if and only if $n = p$ or $2p$ for some prime $p \equiv \pm 3 (mod\ 8)$.*

4. Results on Direct Product of Rings

In this section, we obtain some results regarding the graph-theoretical properties of $\mathbb{S}q(R)$, when R is a direct product of (finitely many) rings. First, we have the following result.

Theorem 4.1. *If R_1 and R_2 are rings with unity such that both $\mathbb{S}q(R_1)$ and $\mathbb{S}q(R_2)$ are connected, then $\mathbb{S}q(R_1 \times R_2)$ is connected.*

Proof. We know that $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is connected (by Theorem 3.6). Clearly, $\mathbb{Z}_2$ is the only ring of order 2 with unity. Now we assume that $|R_1| > 2$, without loss of generality. In $R_1 \times R_2$, let S_1 denote the set of elements of the form $(a, 0)$ (where $a \neq 0$), and S_2 denote the set of elements of the form $(0, b)$, where $b \neq 0$. Consider any two vertices $(a_1, 0)$ and $(a_2, 0)$ from the set S_1. Since $\mathbb{S}q(R_1)$ is connected, we have a walk between a_1 and a_2. If $a_1 \leftrightarrow d_1 \leftrightarrow \cdots \leftrightarrow d_r \leftrightarrow a_2$ is such a walk, then we have a walk $(a_1, 0) \leftrightarrow (d_1, 0) \leftrightarrow \cdots \leftrightarrow (d_r, 0) \leftrightarrow (a_2, 0)$ in $\mathbb{S}q(R_1 \times R_2)$. Similarly there is walk between any two vertices from the set S_2, since $\mathbb{S}q(R_2)$ is connected. Again, $(1, 0) \leftrightarrow (0, 1)$ (we denote the unity of both the rings by 1). So vertices in the set $S_1 \cup S_2$ are lying in the same component in $\mathbb{S}q(R_1 \times R_2)$. Now consider a vertex of the form (a, b), where $a \neq 0, b \neq 0$. Now since $\mathbb{S}q(R_1)$ is connected and $|R_1| > 2$, a has an adjacent vertex in $\mathbb{S}q(R_1)$, say t. Again, since $\mathbb{S}q(R_2)$ is connected, we have that for any $b \neq 1$, there is a walk between b and 1 in $\mathbb{S}q(R_2)$, say $b \leftrightarrow b_1 \leftrightarrow \cdots \leftrightarrow b_s \leftrightarrow 1$. If s is odd then we have a walk $(a, b) \leftrightarrow (t, b_1) \leftrightarrow \cdots \leftrightarrow (t, b_s) \leftrightarrow (a, 1) \leftrightarrow (t, 0)$, and if s is even then we have a walk $(a, b) \leftrightarrow (t, b_1) \leftrightarrow \cdots \leftrightarrow (a, b_s) \leftrightarrow (t, 1) \leftrightarrow (a, 0)$ in $\mathbb{S}q(R_1 \times R_2)$. Again, if $b = 1$, then $(a, b) \leftrightarrow (t, 0)$. So there is a walk from any vertex to a vertex belonging to the set S_1. Since the vertices lying in $S_1 \cup S_2$ are in the same component, $\mathbb{S}q(R_1 \times R_2)$ is connected. ∎

Corollary 4.2. *If $R_1, R_2, \ldots, R_k$ are rings with unity such that $\mathbb{S}q(R_i)$ is connected for all $i = 1, 2, \ldots, k$, then $\mathbb{S}q(R_1 \times R_2 \times \cdots \times R_k)$ is connected.*

Proof. We prove the result by the principle of Mathematical Induction. First, let $k = 2$. Then the result follows from Theorem 4.1. Now, let the result be true for $k = m$. Now $R_1 \times R_2 \times \cdots \times R_m \times R_{m+1} \cong S \times R_{m+1}$, where $S = R_1 \times R_2 \times \cdots \times R_m$. Since $\mathbb{S}q(R_{m+1})$ and $\mathbb{S}q(S)$ are both connected (note that $|S| > 2$), we have by Theorem 4.1 that $\mathbb{S}q(S \times R_{m+1})$, i.e., $\mathbb{S}q(R_1 \times R_2 \times \cdots \times R_m \times R_{m+1})$ is connected. Hence the statement is true for $k = m + 1$. So by the principle of mathematical induction, the result follows. ∎

In the above corollary, we have seen that for commutative rings (with 1) $R_1, R_2, \ldots, R_k$, a sufficient condition for $\mathbb{S}q(R_1 \times R_2 \times \cdots \times R_k)$ to be connected is that $\mathbb{S}q(R_i)$ is connected for all $i = 1, 2, \ldots, k$. However, it is not a necessary condition, as shown in the following proposition.

Proposition 4.3. $\mathbb{S}q(\mathbb{Z}_5 \times \mathbb{Z}_5)$ *is connected.*

Proof. In $\mathbb{S}q(\mathbb{Z}_5 \times \mathbb{Z}_5)$, we denote the set of vertices $\{(\bar{y}, \bar{t}) \mid t = 0, 1, \ldots, 4\}$ by S_y, where $y \in \{1, 2, 3, 4\}$. Also, we denote the set $\{(\bar{0}, \bar{t}) \mid t = 1, \ldots, 4\}$ by S_0.

So S_1, S_2, S_3, S_4 have 5 elements each, whereas S_0 contains 4 elements. It is easy to see that in $\mathbb{S}q(\mathbb{Z}_5 \times \mathbb{Z}_5)$, the subgraph induced by the vertices in S_2 and the subgraph induced by the vertices in S_3 both contain a P_5 (viz., the paths $(x, \bar{0}) \leftrightarrow (x, \bar{1}) \leftrightarrow (x, \bar{3}) \leftrightarrow (x, \bar{2}) \leftrightarrow (x, \bar{4})$ for $x = \bar{2}$ and $x = \bar{3}$, respectively). Since $|S_2| = 5$, this implies that the vertices of S_2 are in a single component. For the same reason, the vertices of S_3 are in a single component. Now $(\bar{1}, \bar{r})$ is adjacent to $(\bar{3}, \overline{1-r})$ for each $r = 0, 1, \ldots, 4$. So the vertices in $S_3 \cup S_1$ are in the same component. Similarly, noting that $(\bar{4}, \overline{1-r}) \leftrightarrow (\bar{2}, \bar{r})$ for $r = 0, 1, \ldots, 4$, we have that the vertices in $S_2 \cup S_4$ are in a single component. Now we note that $(\bar{0}, \bar{1}) \leftrightarrow (\bar{1}, \bar{0}), (\bar{0}, \bar{2}) \leftrightarrow (\bar{1}, \bar{3}), (\bar{0}, \bar{3}) \leftrightarrow (\bar{1}, \bar{1})$, and $(\bar{0}, \bar{4}) \leftrightarrow (\bar{1}, \bar{0})$. Thus the vertices in $S_0 \cup S_1 \cup S_3$ are in a single component. We have already seen that vertices in $S_2 \cup S_4$ are in a single component. So we have at most two components. Now, we note that $(\bar{2}, \bar{2})$ of S_2 is adjacent to $(\bar{3}, \bar{2})$ in S_3. Hence, $\mathbb{S}q(\mathbb{Z}_5 \times \mathbb{Z}_5)$ is connected. ∎

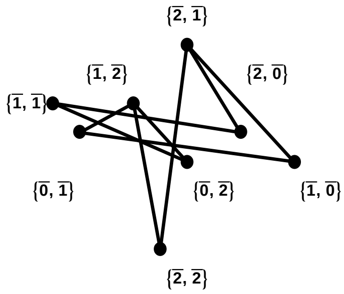

Figure 10: $\mathbb{S}q(\mathbb{Z}_3 \times \mathbb{Z}_3)$

Also, Figure 10 shows that $\mathbb{S}q(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is connected. So the converse of Corollary 4.2 is not true, since $\mathbb{S}q(\mathbb{Z}_3)$ and $\mathbb{S}q(\mathbb{Z}_5)$ are disconnected (cf. Theorem 3.4). Now we consider the graphs $\mathbb{S}q(\mathbb{Z}_n \times R)$ for $n = 3, 5$, in general.

Theorem 4.4. *If R is a ring with unity such that $\mathbb{S}q(R)$ is connected, then $\mathbb{S}q(\mathbb{Z}_3 \times R)$ and $\mathbb{S}q(\mathbb{Z}_5 \times R)$ are connected.*

Proof. If $|R| = 2$, then $R \cong \mathbb{Z}_2$. Now $\mathbb{S}q(\mathbb{Z}_3 \times \mathbb{Z}_2)$ and $\mathbb{S}q(\mathbb{Z}_5 \times \mathbb{Z}_2)$ (being isomorphic to $\mathbb{S}q(\mathbb{Z}_6)$ and $\mathbb{S}q(\mathbb{Z}_{10})$, respectively) are connected by Theorem 3.4. So we assume $|R| > 2$. First, we consider the graph $\mathbb{S}q(\mathbb{Z}_3 \times R)$. Proceeding similarly to what we did in Theorem 4.1, it is easy to show that there is a path between any two vertices of the form $(\bar{0}, a)$ (where $a \neq 0$). Now considering a vertex of the form $(\bar{1}, a)$ where $a \neq 0$, we have that $(\bar{1}, a) \leftrightarrow (\bar{0}, b)$, where b is any adjacent vertex of a in $\mathbb{S}q(R)$. Also, $(\bar{1}, 0) \leftrightarrow (\bar{0}, 1)$. Again, for a vertex of the form $(\bar{2}, a)$ where $a \neq 0$, we have $(\bar{2}, a) \leftrightarrow (\bar{1}, b) \leftrightarrow (\bar{0}, a)$. Also $(\bar{2}, 0) \leftrightarrow (\bar{2}, 1) \leftrightarrow (\bar{1}, 0) \leftrightarrow (\bar{0}, 1)$. Hence, $\mathbb{S}q(\mathbb{Z}_3 \times R)$ is connected.

Next, we consider the graph $\mathbb{S}q(\mathbb{Z}_5 \times R)$. As before, the vertices of the form $(\bar{0}, a)$ (where $a \neq 0$) are in the same component. Now consider a vertex $(\bar{1}, a)$. If $a = 0$, then $(\bar{1}, a) \leftrightarrow (\bar{0}, 1)$. If $a \neq 0$, then $(\bar{1}, a) \leftrightarrow (\bar{0}, b)$, where b is any adjacent vertex of a in $\mathbb{S}q(R)$. Next, we consider a vertex $(\bar{2}, a)$. If $a = 0$, then we have a path $(\bar{2}, 0) \leftrightarrow (\bar{3}, 1) \leftrightarrow (\bar{1}, 0) \leftrightarrow (\bar{0}, 1)$. If $a \neq 0$, then we have a path $(\bar{2}, a) \leftrightarrow (\bar{3}, b) \leftrightarrow (\bar{1}, a) \leftrightarrow (\bar{0}, b)$, where b is any adjacent vertex of a in $\mathbb{S}q(R)$. The last two paths, in fact, also show that there is a path from any vertex of the form $(\bar{3}, a)$ to a vertex of the form $(\bar{0}, b)$ if $a \neq 0$. Again, $(\bar{3}, \bar{0}) \leftrightarrow (\bar{1}, \bar{0}) \leftrightarrow (\bar{0}, \bar{1})$. Lastly, we consider a vertex of the form $(\bar{4}, a)$. If $a = 0$, then $(\bar{4}, 0) \leftrightarrow (\bar{0}, 1)$. If $a \neq 0$, then we have that $(\bar{4}, a) \leftrightarrow (\bar{0}, b)$, where b is any adjacent vertex of a in $\mathbb{S}q(R)$. So $\mathbb{S}q(\mathbb{Z}_5 \times R)$ is connected. ∎

Next, we consider the direct products of rings of the form $\mathbb{Z}_n$ in general.

Theorem 4.5. *$\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n)$ is connected for any integers $m, n > 1$.*

Proof. First, let $m = 2$. We have seen that $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is connected (cf. Theorem 3.6). Also, $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_3)$ and $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_5)$ are connected (being isomorphic to $\mathbb{S}q(\mathbb{Z}_6)$ and $\mathbb{S}q(\mathbb{Z}_{10})$, respectively). For $n > 5$, $\mathbb{S}q(\mathbb{Z}_n)$ is connected, and hence $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_n)$ is connected by Theorem 4.1. Now, let $m = 3$. $\mathbb{S}q(\mathbb{Z}_3 \times \mathbb{Z}_2)$, $\mathbb{S}q(\mathbb{Z}_3 \times \mathbb{Z}_4)$, and $\mathbb{S}q(\mathbb{Z}_3 \times \mathbb{Z}_5)$ are connected as they are isomorphic to $\mathbb{S}q(\mathbb{Z}_6)$, $\mathbb{S}q(\mathbb{Z}_{12})$, and $\mathbb{S}q(\mathbb{Z}_{15})$ respectively. From the Figure 10, it can be seen that $\mathbb{S}q(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is connected. Again, for $n > 5$, the graph $\mathbb{S}q(\mathbb{Z}_3 \times \mathbb{Z}_n)$ is connected by Theorem 4.4. Now let us assume that $m > 3$. We can also assume $n > 3$, because $\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n) \cong \mathbb{S}q(\mathbb{Z}_n \times \mathbb{Z}_m)$ and we have already shown that $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_r)$ and $\mathbb{S}q(\mathbb{Z}_3 \times \mathbb{Z}_r)$ are connected for any $r > 1$. By Proposition 4.3, $\mathbb{S}q(\mathbb{Z}_5 \times \mathbb{Z}_5)$ is connected. If $n = 4$ or $n > 5$, then $\mathbb{S}q(\mathbb{Z}_5 \times \mathbb{Z}_n)$ is connected by Theorem 4.4. Finally, if $m, n > 5$, then $\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n)$ is connected by Theorem 4.1. So having considered all possible cases, we reach the result that $\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n)$ is connected for $m, n > 1$. ∎

Corollary 4.6. *$\mathbb{S}q(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k})$ is connected, where $k > 1$.*

Proof. If k is even, then $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k} \cong (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \times (\mathbb{Z}_{m_3} \times \mathbb{Z}_{m_4}) \times \cdots \times (\mathbb{Z}_{m_{k-1}} \times \mathbb{Z}_{m_k})$. So the result follows immediately from Theorem 4.5 and Corollary 4.2. Let k be odd. Then $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k} \cong \mathbb{Z}_{m_1} \times R$, where $R \cong \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$. Since R is a direct product of even number of $\mathbb{Z}_n$ rings, $\mathbb{S}q(R)$ is connected as we have just shown. Now if $m_1 \in \{3, 5\}$, then $\mathbb{S}q(\mathbb{Z}_{m_1} \times R)$ is connected by Theorem 4.4. Otherwise, $\mathbb{S}q(\mathbb{Z}_{m_1} \times R)$ is connected by Theorem 4.1. Hence the result. ∎

Next, we move to direct products of finite fields.

Theorem 4.7. *If $F_1, F_2, \dots, F_k$ are finite fields (where $k > 1$), then $\mathbb{S}q(F_1 \times F_2 \times \cdots \times F_k)$ is connected.*

Proof. We prove this by the principle of Mathematical induction. We recall that for any finite field F, $\mathbb{S}q(F)$ is connected if and only if $F \ncong \mathbb{Z}_3, \mathbb{Z}_5$ (cf. Theorem 3.5). First, let $k = 2$. If $F_1, F_2 \ncong \mathbb{Z}_3, \mathbb{Z}_5$, then $\mathbb{S}q(F_1 \times F_2)$ is connected by Theorem 4.1. If exactly one of F_1 and F_2 is isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_5$, then $\mathbb{S}q(F_1 \times F_2)$ is connected by Theorem 4.4. Also, $\mathbb{S}q(\mathbb{Z}_p \times \mathbb{Z}_q)$ (where p, q may not be distinct) is connected for $p, q \in \{3, 5\}$ by Theorem 4.5. So $\mathbb{S}q(F_1 \times F_2)$ is always connected. Now let the statement be true for $k = m$. We consider $\mathbb{S}q(F_1 \times F_2 \times \cdots \times F_m \times F_{m+1})$. Clearly, $F_1 \times F_2 \times \cdots \times F_m \times F_{m+1} \cong R \times F_{m+1}$, where $R \cong F_1 \times F_2 \times \cdots \times F_m$. From the induction hypothesis, $\mathbb{S}q(R)$ is connected. If $F_{m+1} \cong \mathbb{Z}_3$ or $\mathbb{Z}_5$, then $\mathbb{S}q(R \times \mathbb{Z}_{m+1})$ is connected by Theorem 4.4. Otherwise, $\mathbb{S}q(R \times \mathbb{Z}_{m+1})$ is connected by Theorem 4.1. So the statement is true for $k = m + 1$. Hence, by the principle of Mathematical Induction, the result follows. ■

Corollary 4.8. *If R is a finite commutative Jacobson semisimple ring with 1, then $\mathbb{S}q(R)$ is connected.*

Proof. It is known that any finite commutative Jacobson semisimple ring with 1 is a finite direct product of finite fields. Hence the result follows from Theorem 4.7. ■

Remark 4.9. Note that the converse of Corollary 4.8 is not true. In other words, if R is a finite commutative ring and $\mathbb{S}q(R)$ is connected then that does not imply that R is Jacobson semisimple. In the ring $\mathbb{Z}_{12}$, $I = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\}$ and $J = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\}$ are the only maximal ideals of $\mathbb{Z}_{12}$. Now $I \cap J = \{\overline{0}, \overline{6}\} \neq \{\overline{0}\}$. So the ring $\mathbb{Z}_{12}$ is not J-semisimple. However, $\mathbb{S}q(\mathbb{Z}_{12})$ is connected.

Now we look at the regularity of $\mathbb{S}q(R)$ taken over direct products of $\mathbb{Z}_n$ rings.

Theorem 4.10. *$\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n)$ is regular if and only if either $m = n = 2$ or (without loss of generality) $m = 2, n = p$ or $2p$, for any prime $p \equiv \pm 3$ mod 8.*

Proof. We recall that for any ring R, the graph $\mathbb{S}q(R)$ is regular if and only if for each $v \in R - \{0\}$, exactly one of v and $2v$ is a square element (cf. Proposition 3.9). Then we consider the following cases.

Case I: Let $m = n = 2$. Figure 3 shows that $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong K_3$, i.e., it is regular.

Case II: Let $m = 2, n > 2$ and $\mathbb{S}q(\mathbb{Z}_n)$ be regular. So $n = p$ or $2p$ for some prime $p \equiv \pm 3$ *mod* 8 (cf. Theorem 3.10). Consider a non-zero element (a, b) in $\mathbb{Z}_m \times \mathbb{Z}_n$. Here $a = \overline{0}$ or $\overline{1}$. We note that $(\overline{1}, \overline{0})$ is a square element whereas $2 \cdot (\overline{1}, \overline{0})$, i.e., $(\overline{0}, \overline{0})$ is not, since n is squarefree. Now let (a, b) be a square element distinct from $(\overline{1}, \overline{0})$. So b is a square in $\mathbb{Z}_n$. This gives that $2b$ is a non-square in $\mathbb{Z}_n$ (since $\mathbb{S}q(\mathbb{Z}_n)$ is regular), and consequently, $2 \cdot (a, b)$ i.e., $(\overline{0}, 2b)$ is a non-square in $\mathbb{Z}_m \times \mathbb{Z}_n$. Again, let (a, b) be non-square. So b is a non-square in $\mathbb{Z}_n$ and hence $2b$ is a square in $\mathbb{Z}_n$. As a result, $2 \cdot (a, b)$, i.e., $(\bar{0}, 2b)$ is a square element in $\mathbb{Z}_m \times \mathbb{Z}_n$. So in this case, for every non-zero v, exactly one of v and $2v$ is a square in $\mathbb{Z}_m \times \mathbb{Z}_n$. So $\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n)$ is regular.

Case III: Let $m = 2, n > 2$ and $\mathbb{S}q(\mathbb{Z}_n)$ be not regular. So in $\mathbb{Z}_n$, we either have some non-zero a such that both $a, 2a$ are squares in $\mathbb{Z}_n$, or we have some non-zero b such that both $b, 2b$ are not squares in $\mathbb{Z}_n$. In the first case, both $(\bar{0}, a)$ and $2 \cdot (\bar{0}, a)$, (i.e., $(\bar{0}, 2a)$) are squares in $\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n)$. In the second case, both $(\bar{0}, a)$ and $2 \cdot (\bar{0}, a)$ (i.e., $(\bar{0}, 2a)$) are non-squares in $\mathbb{Z}_m \times \mathbb{Z}_n$. So in both the cases, $\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n)$ is not regular.

Case IV: Let $m > 2, n > 2$ and $\mathbb{S}q(\mathbb{Z}_n)$ be not regular (without loss of generality). Then arguing as we did in Case III, it is easy to see that $\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n)$ is not regular.

Case V: Let $m > 2, n > 2$ and both $\mathbb{S}q(\mathbb{Z}_m)$ and $\mathbb{S}q(\mathbb{Z}_n)$ be regular. If t is a non-zero non-square in $\mathbb{Z}_m$ (such a t exists since $m > 2$), then we have that both $(t, \bar{1})$ and $2 \cdot (t, \bar{1})$ (i.e., $(2t, \bar{2})$) are non-squares in $\mathbb{Z}_m \times \mathbb{Z}_n$ (note that $\bar{2}$ is non-square in $\mathbb{Z}_n$ since $\mathbb{S}q(\mathbb{Z}_n)$ is regular. In fact $\bar{2}$ is non-square even when $n = 2$). Hence $\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n)$ is not regular.

Having considered all possible cases (note that $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_n \times \mathbb{Z}_m$), we see that $\mathbb{S}q(\mathbb{Z}_m \times \mathbb{Z}_n)$ is regular only in the first two cases. Hence the result. ∎

Theorem 4.11. $\mathbb{S}q(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k})$ *(where $k > 2$) is regular if and only if $m_i \neq 2$ for at most one i in $\{1, 2, \ldots, k\}$ and that $m_i = p$ or $2p$ for some prime $p \equiv \pm 3$ mod 8.*

Proof. If possible, let there exist distinct i, j in $\{1, 2, \ldots, k\}$ such that $m_i, m_j \neq 2$. Then we can rearrange the order in the direct product to make $m_1, m_2 \neq 2$. From the proof of Theorem 4.10, we have that in $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$, we have either a non-zero element (a, b) such that both (a, b) and $(2a, 2b)$ are squares, or a non-zero element (a, b) such that both (a, b) and $(2a, 2b)$ are non-squares. In the first case, both $(a, b, \bar{0}, \bar{0}, \cdots, \bar{0})$ and $(2a, 2b, \bar{0}, \bar{0}, \cdots, \bar{0})$ are squares, and in the second case both $(a, b, \bar{0}, \bar{0}, \cdots, \bar{0})$ and $(2a, 2b, \bar{0}, \bar{0}, \cdots, \bar{0})$ are non-squares in $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$. So we can have at most one i in $\{1, 2, \ldots, k\}$ such that $m_i \neq 2$. Now consider $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_m)$, where $m \neq 2$. If $\mathbb{S}q(\mathbb{Z}_m)$ is not regular, then we can show in exactly the same way that $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_m)$ is not regular. Again, let $\mathbb{S}q(\mathbb{Z}_m)$ be regular, i.e., $m = p$ or $2p$ for some prime $p \equiv \pm 3$ *mod* 8. Consider a nonzero element $s = (a_1, a_2, \cdots, a_{m-1}, b)$ in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_m$. Clearly, $2a_j = \bar{0}$ for all $j = 1, 2, \ldots, m-1$. If s is a square then $b = \bar{0}$ or b is a square element (note that $\bar{0}$ is not a square in $\mathbb{Z}_m$). In either case, $2s = (2a_1, 2a_2, \cdots, 2a_{m-1}, 2b)$ is a non-square (since $\mathbb{S}q(\mathbb{Z}_m)$ is regular) in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_m$. Again, if s is a non-square then b must be a non-zero non-square (since s is non-zero) in $\mathbb{Z}_m$. Hence, $2b$ is a square in $\mathbb{Z}_m$, and consequently, $2s$ (i.e., $(2a_1, 2a_2, \cdots, 2a_{m-1}, 2b)$) is a square in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_m$. So for each non-zero v, exactly one of v and $2v$ is a square element in this case. Finally, we know that $\mathbb{S}q(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2)$ is complete, and hence, regular.

Hence the result. ■

5. Results on Infinite Rings

In this section, we look at some properties of $\mathbb{S}q(R)$ when R is an infinite ring. The ring $\mathbb{Z}$ is of special interest in this regard, since it is contained in any infinite commutative ring with 1. We first have the folowing result.

Theorem 5.1. $\mathbb{S}q(\mathbb{Z})$ *is connected.*

Proof. We show that for any vertex $v \neq -1$, there is a walk from v to -1 in $\mathbb{S}q(\mathbb{Z})$. First, suppose that v is a square element. So $v = x^2$ for some $x \in \mathbb{Z} - \{0\}$. For $x^2 = 1$, we have a path $1 \leftrightarrow 8 \leftrightarrow -4 \leftrightarrow 5 \leftrightarrow -1$. Next, suppose that $x^2 \neq 1$. Then we have a walk $v \leftrightarrow 1 - 2x \leftrightarrow 2x \leftrightarrow 1 + x^2 \leftrightarrow -1$. Again, let v be a non-square. Suppose that v is a positive integer. First, let v be not of the form $4k+2$. Then $v = x^2 - y^2$ for some $x, y \in \mathbb{Z}$. Now $x \neq 0$ as v is positive, and $y \neq 0$ as v is not a square. If $y^2 = 1$, we have a path $v \leftrightarrow 1 \leftrightarrow 8 \leftrightarrow -4 \leftrightarrow 5 \leftrightarrow -1$. This also shows that there exists a path from v to -1 if $v = 5, 8$. Suppose that $y^2 \neq 1$ and $v \neq 5, 8$. Then we have a walk $v \leftrightarrow y^2 \leftrightarrow 1-2y \leftrightarrow 2y \leftrightarrow 1+y^2 \leftrightarrow -1$. Again, let v be a positive integer of the form $4k + 2$ for some positive integer k. Then we have a walk $v \leftrightarrow 4k^2 - 1 \leftrightarrow 1 \leftrightarrow 8 \leftrightarrow -4 \leftrightarrow 5 \leftrightarrow -1$. Finally, let v be a negative integer and $v \neq -1$. Clearly, $v \leftrightarrow 1 - v$. Now since $1 - v$ is a positive integer, there is a walk (say, W) from $1 - v$ to -1, as per our earlier arguments. So we have a walk $v \leftrightarrow 1 - v \leftrightarrow W \leftrightarrow -1$. Hence, there is a walk from any vertex v $(\neq -1)$ to the vertex -1 in $\mathbb{S}q(\mathbb{Z})$. So $\mathbb{S}q(\mathbb{Z})$ is connected. ■

Remark 5.2. Note that $\mathbb{S}q(\mathbb{Z})$ is not complete. Square elements in $\mathbb{Z}$ are the squares of all natural numbers. Hence, 3 is not a square element. So 1 is not adjacent to 2 in $\mathbb{S}q(\mathbb{Z})$.

Corollary 5.3. (i) $\mathbb{S}q(\mathbb{Z} \times R)$ *is connected for any commutative ring R (with 1) such that $\mathbb{S}q(R)$ is connected.*

(ii) $\mathbb{S}q(\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{k \text{ times}})$ *is connected for any $k \in \mathbb{N}$.*

Proof. (i) This can be proved exactly in the way as we did in Theorem 4.1.
(ii) This follows by successive application of (i). ■

The last corollary gives many infinite rings for which $\mathbb{S}q(R)$ is connected. However, $\mathbb{S}q(R)$ is not connected for $R \cong \mathbb{Z}[x]$, as we show in the following proposition.

Proposition 5.4. $\mathbb{S}q(\mathbb{Z}[x])$ *is not connected.*

Proof. We first note that in $\mathbb{Z}[x]$, the co-efficient of x in the square of any polynomial is either 0 or an even non-zero positive integer. So $1 + x$ is not a square element in $\mathbb{Z}[x]$ and consequently, $1 \not\leftrightarrow x$ in $\mathbb{S}q(\mathbb{Z}[x])$. We show that there is no path between 1 and x in $\mathbb{S}q(\mathbb{Z}[x])$. If possible, let $x \leftrightarrow f_1 \leftrightarrow f_2 \leftrightarrow \cdots \leftrightarrow f_k \leftrightarrow 1$ be a path from x to 1, where the f_i's are polynomial in $\mathbb{Z}[x]$, for $i = 1, 2, \ldots, k$. So we have that $x + f_1 = h_1^2$; $f_1 + f_2 = h_2^2; \ldots;$ $f_{k-1} + f_k = h_k^2$; $f_k + 1 = h_{k+1}^2$ for some non-zero polynomials $h_1, h_2, \ldots, h_k$. Hence, we have that $h_1^2 - h_2^2 + \cdots + (-1)^{k-1} h_k^2 + (-1)^k h_{k+1}^2 = x + (-1)^k$. Now since the coefficient of x is even in h_t^2 for $t = 1, 2, \ldots, k+1$, we have that the coefficient of x in the Left Hand Side of the last equation is also even. This is a contradiction since Right Hand Side of the equation is $x + (-1)^k$. Hence, we cannot have a path between x and 1. So $\mathbb{S}q(\mathbb{Z}[x])$ is not connected. ■

Next, we have some results regarding cycles in $\mathbb{S}q(\mathbb{Z})$. Note that if R is an infinite ring with unity 1 and characteristic 0, then every cycle in $\mathbb{S}q(\mathbb{Z})$ corresponds to a cycle in $\mathbb{S}q(R)$ also.

Theorem 5.5. *For all $y \in \mathbb{Z} - \{0, 1\}$, there exists a cycle C in $\mathbb{S}q(\mathbb{Z})$ such that y and 1 are vertices in C.*

Proof. Let $m \in \mathbb{Z} - \{0, 1\}$. It is easy to check that if $m \notin \{-1, 2, -2, 4, -4\}$, then $1 \leftrightarrow 3 \leftrightarrow m^2 + 4m + 1 \leftrightarrow -2m \leftrightarrow 2m + 1 \leftrightarrow m^2 \leftrightarrow 4m + 4 \leftrightarrow -4m \leftrightarrow m^2 + 4 \leftrightarrow -4 \leftrightarrow 8 \leftrightarrow 1$ is a cycle in $\mathbb{S}q(\mathbb{Z})$. (For $m = -1, \pm 2, \pm 4$, it may fail to be a cycle as expressions of m may give 0 or repeated values). We call this cycle C_m. Now we take any $y \neq 0, 1$ and look for a cycle containing both y and 1. If $y \in \{-1, 2, 3, -4, 5, 7, -8, 8, 9\}$ then we have such a cycle $1 \leftrightarrow 8 \leftrightarrow -4 \leftrightarrow 5 \leftrightarrow -1 \leftrightarrow 2 \leftrightarrow 7 \leftrightarrow 9 \leftrightarrow -8 \leftrightarrow 33 \leftrightarrow 3 \leftrightarrow 1$. If $y \in \{-2, -3, 4, -5, 6, -6, -7\}$ then we have such a cycle $1 \leftrightarrow 8 \leftrightarrow -7 \leftrightarrow 11 \leftrightarrow -2 \leftrightarrow 6 \leftrightarrow -5 \leftrightarrow 21 \leftrightarrow 4 \leftrightarrow -3 \leftrightarrow 7 \leftrightarrow -6 \leftrightarrow 15 \leftrightarrow 1$. Again, if $y = -9$, we have $y = 2m + 1$ where $m = -5$. So C_{-5} is a cycle containing both y and 1 as vertices. So we have considered all $y \neq 0, 1$ such that $|y| \leq 9$. Finally, we see that if $|y| > 9$, then y can be expressed as $2m + 1$ or $-2m$ (according as y is odd or even) where $|m| \geq 5$. So for any y with $|y| > 9$, we have some suitable value of m such that C_m is a cycle containing both y and 1. Hence the result. ■

Corollary 5.6. *If R is an infinite ring with 1 such that $Char(R) = 0$, then there are infinitely many cycles in $\mathbb{S}q(R)$.*

Proof. First, we show that $\mathbb{S}q(\mathbb{Z})$ contains infinitely many cycles. Now from Theorem 5.5, we have that for each $y \in \mathbb{Z} - \{0, 1\}$, there is a finite cycle C_y containing both y and 1. If $\mathbb{S}q(\mathbb{Z})$ contains only finitely many cycles, then the number of such cycles C_y's will also be finite (note that it is possible since we can have the same cycle C_y for multiple values of y). Now since each such cycle contains only finitely many integers (as the length of the cycle is finite), we have that only finitely many integers are involved in $\{C_y \mid y \in \mathbb{Z} - \{0, 1\}\}$. This contradicts that for each integer $y \neq 0, 1$, there is a cycle in $\mathbb{S}q(\mathbb{Z})$ such that the cycle contains both y and 1. Hence $\mathbb{S}q(\mathbb{Z})$ must contain infinitely many cycles. Now let R be an inifnite ring with 1 and $Char(R) = 0$. Then we have that $\mathbb{Z} \subset R$, i.e., $\mathbb{S}q(\mathbb{Z})$ is a subgraph of $\mathbb{S}q(R)$. Hence, $\mathbb{S}q(R)$ contains infinitely many cycles. ∎

Remark 5.7. We note that for any $n \in \mathbb{N}$, $1^3 + 2^3 + 3^3 + \cdots + n^3 (= (\frac{n(n+1)}{2})^2)$ is a square element. Also we have $2^3 + 4^3 + 6^3 + \cdots + (2n-2)^3 + (2n)^3 = 8(1^3 + 2^3 + \cdots + n^3) = 8(\frac{n(n+1)}{2})^2$.

Theorem 5.8. *Let R be an infinite ring with 1 such that $Char(R) = 0$. Then for any $n > 3$, $\mathbb{S}q(R)$ has a cycle of length greater than n.*

Proof. Consider any $n > 3$. Clearly, it suffices to show that $\mathbb{S}q(\mathbb{Z})$ contains a cycle of length greater than n. Taking into consideration the result discussed in Remark 5.7, we see that $\mathbb{S}q(\mathbb{Z})$ has a closed walk $-1 \leftrightarrow 5 \leftrightarrow -4 \leftrightarrow 8 \leftrightarrow 1^3+3^3 \leftrightarrow 2^3+4^3 \leftrightarrow 1^3+3^3+5^3 \leftrightarrow 2^3+4^3+6^3 \leftrightarrow \cdots \leftrightarrow 1^3+3^3+5^3+\cdots+(2n^2-1)^3 \leftrightarrow 2^3+4^3+\cdots+(2n^2)^3 \leftrightarrow 1^3+2^3+3^3+\cdots+(n^2-1)^3+n^6 \leftrightarrow -n^6 \leftrightarrow 1+n^6 \leftrightarrow -1$. Noting that the power of 2 in $1^3+3^3+\cdots+(2k+1)^3$ (i.e., in $(k+1)^2(2k^2+4k+1)$) is even (for any k) and the power of 2 in $2^3+4^3+\cdots+(2l)^3$ is odd (for any l), it is easy to see that the closed walk described indeed gives a cycle. Clearly, the length of the cycle is greater than n. So in $\mathbb{S}q(\mathbb{Z})$ (and hence, in $\mathbb{S}q(R)$ for any infinite ring R with 1 and characteristic 0), there is a cycle of length greater than n for any $n > 3$. ∎

Next, we find the girth of $\mathbb{S}q(R)$ for any infinite ring R with unity and characteristic 0.

Theorem 5.9. *If R is an infinite ring with 1 and $Char(R) = 0$, then $girth(\mathbb{S}q(R)) = 3$.*

Proof. Since R is an infinite commutative ring with 1 and $Char(R) = 0$, we have that $\mathbb{Z} \subset R$. So we have the 3-cycle $6 \leftrightarrow 3 \leftrightarrow -2 \leftrightarrow 6$ in $\mathbb{S}q(R)$. ∎

Corollary 5.10. *If R is an infinite ring with 1 and $Char(R) = 0$, then $\mathbb{S}q(R)$ is never bipartite, and hence is never a tree.*

Proposition 5.11. $\mathbb{S}q(\mathbb{Z})$ *is not a perfect graph.*

Proof. This follows immediately from the fact that $\mathbb{S}q(\mathbb{Z})$ has an induced subgraph which is isomorphic to C_5, viz., the subgraph induced by the vertices $\{4, 5, -1, 37, 12\}$. ∎

Now we look at some properties of $\mathbb{S}q(F)$, where F is any field.

Proposition 5.12. *If a field F has only finite number of square elements, then F must be a finite field.*

Proof. Let a field F have n square elements for some $n \in \mathbb{N}$. If possible, let F be infinite. Now since F is a field, $x^2 = y^2$ implies that either $x = y$ or $x = -y$. Hence, each square element in F is the square of at most two elements in F. So $F - \{0\}$ can have at most $2n$ elements. This contradicts that F is infinite. So F must be a finite field. ∎

We give a necessary condition for $\mathbb{S}q(F)$ to be complete.

Theorem 5.13. *If F is a field, then $\mathbb{S}q(F)$ is complete only if $Char(F) = 2$.*

Proof. Let F be a field and suppose $\mathbb{S}q(F)$ is complete. If possible, let $Char(F) \neq 2$. Then $1, -1$ are distinct elements of F. Hence, 1 and -1 are adjacent vertices in $\mathbb{S}q(F)$. So there exists some $x \neq 0$ such that $1 + (-1) = x^2$. This gives that $x^2 = 0$, which is a contradiction since F is a field. So we must have that $Char(F) = 2$. ∎

However, the converse of Theorem 5.13 is not always true as we shall show shortly. Before that, we show that for a field F with $Char(F) = 2$, $\mathbb{S}q(F)$ is either disconnected or complete.

Theorem 5.14. *If F is a field with characteristic 2 and $\mathbb{S}q(F)$ is connected, then all non-zero elements of F are square elements and $\mathbb{S}q(F)$ is complete.*

Proof. Let F be a field with $Char(F) = 2$ and suppose that $\mathbb{S}q(F)$ is connected. Let $S = \{t \mid t \text{ is a square element in } F\} \cup \{0\}$. If possible, let $F - S \neq \phi$. Since $\mathbb{S}q(F)$ is connected, we must have that at least one pair of adjacent vertices of the form $\{a, r^2\}$ such that $a \in F - S$ and $r^2 \in S$. So $a + r^2 = m^2$ for some $m \neq 0$. Since $Char(F) = 2$, this gives that $a = m^2 - r^2 = m^2 + r^2 = (m + r)^2$. Now since $a \neq 0$, and F is a field, we must have that $m+r \neq 0$. Consequently, a is a square element, which is a contradiction. Hence, $F \cong S$. So every non-zero element in F is a square element. Now since $Char(F) = 2$, we have that $y = -y$ for all $y \in F$. Now consider distinct $a, b \in F - \{0\}$. Clearly, $a = c^2$ and $b = d^2$ for some $c, d \neq 0$. So $a + b = c^2 + d^2 = (c + d)^2$. Since $a \neq b$, we have that $c \neq d$, i.e., $c + d \neq 0$ as $Char(F) = 2$. So $a \leftrightarrow b$. Hence $\mathbb{S}q(F)$ is complete. ∎

Now we look for a field F of characteristic 2 such that $\mathbb{S}q(F)$ is not complete. First, we have the following proposition.

Proposition 5.15. *$\mathbb{S}q(\mathbb{Z}_2[x])$ is not connected. However, the vertices corresponding to the square elements induce a complete subgraph in $\mathbb{S}q(\mathbb{Z}_2[x])$.*

Proof. Any polynomial in $\mathbb{Z}_2[x]$ is of the form $x^{m_1} + x^{m_2} + \cdots + x^{m_r}$ or $1 +$

$x^{m_1} + x^{m_2} + \cdots + x^{m_r}$, where $m_i \in \mathbb{N}$ for $i = 1, 2, \ldots, r$ and $r \in \mathbb{N}$. Now since the characteristic of $\mathbb{Z}_2[x]$ is 2, we have that $(x^{m_1} + x^{m_2} + \cdots + x^{m_r})^2 = x^{2m_1} + (x^{m_2} + \cdots + x^{m_r})^2 = \cdots = x^{2m_1} + x^{2m_2} + \cdots + x^{2m_r}$. Similarly, $(1 + x^{m_1} + x^{m_2} + \cdots + x^{m_r})^2 = 1 + x^{2m_1} + x^{2m_2} + \cdots + x^{2m_r}$. Conversely, any polynomial of the form $c + x^{2n_1} + x^{2n_2} + \cdots + x^{2n_r}$ (where $c = 0$ or 1) is the square of the polynomial $c + x^{n_1} + x^{n_2} + \cdots + x^{n_r}$. So the square elements in $\mathbb{Z}_2[x]$ are precisely the polynomials which contain no odd powers of x as a monomial. Now this shows that the sum of any two square element is also a square element, since $a = -a$ for all $a \in \mathbb{Z}_2[x]$. Hence, the subgraph of $\mathbb{S}q(\mathbb{Z}_2[x])$ induced by the square element vertices is a complete subgraph. Now if $\mathbb{S}q(\mathbb{Z}_2[x])$ has to be connected, then we must have at least one pair of adjacent vertices $\{v, w\}$ such that (without loss of generality) v is a square and w is a non-square. Now w contains at least one monomial of the form x^t where t is odd, and v contains no odd powers of x as a monomial. Hence, x^t is a monomial in $v + w$. So $v + w$ is not a square element. Thus, a square element vertex is never adjacent to any non-square element vertex. So $\mathbb{S}q(\mathbb{Z}_2[x])$ is not connected. ∎

Remark 5.16. From the proof of Proposition 5.15, we see that the vertices 1 and x are not adjacent in $\mathbb{S}q(\mathbb{Z}_2[x])$. Now we consider $Q(\mathbb{Z}_2[x])$, i.e., the quotient field of $\mathbb{Z}_2[x]$. Clearly, $Q(\mathbb{Z}_2[x]) = \{\frac{f(x)}{g(x)} \mid f(x), g(x) \in \mathbb{Z}_2[x], g(x) \neq 0\}$. So the square elements in $Q(\mathbb{Z}_2[x])$ are of the form $\frac{a^2}{b^2}$, where $a, b \in \mathbb{Z}_2[x]$ and $b^2 \neq 0$. Thus, $1 + x$ is not a square element in $Q(\mathbb{Z}_2[x])$ also. This shows that $1, x$ are not adjacent in $\mathbb{S}q(Q(\mathbb{Z}_2[x]))$. So although the field $Q(\mathbb{Z}_2[x])$ is of characteristic 2, yet $\mathbb{S}q(Q(\mathbb{Z}_2[x]))$ is not complete. In other words, $Char(F) = 2$ does not imply that $\mathbb{S}q(F)$ is complete. This is different from the case of finite fields since for a finite field F with $Char(F) = 2$, $\mathbb{S}q(F)$ is complete (cf. Theorem 3.7).

We now give a necessary and sufficient condition for $\mathbb{S}q(F)$ to be complete, where F is any finite field.

Theorem 5.17. *Let F be a field. Then $\mathbb{S}q(F)$ is complete if and only if $Char(F) = 2$ and every non-zero element of F is a square element.*

Proof. Let F be a field such that $\mathbb{S}q(F)$ is complete. By Theorem 5.13, we have that $Char(F) = 2$. Since $\mathbb{S}q(F)$ is connected, it follows from Theorem 5.14 that all non-zero elements of F are square elements.

Conversely, let $Char(F) = 2$ and suppose that all non-zero elements of F are square elements. Since $a = -a$ for all $a \in F$, this implies that the sum of any two distinct elements of $F - \{0\}$ is a square element. So $\mathbb{S}q(F)$ is a complete graph. ∎

Remark 5.18. Note that Remark 5.16 gives us an example of an infinite field F such that $\mathbb{S}q(F)$ is not connected. If $\mathbb{S}q(Q(\mathbb{Z}_2[x]))$ was connected, then from Theorem 5.14, we would have that $\mathbb{S}q(Q(\mathbb{Z}_2[x]))$ is complete. However, as we have shown in Remark 5.16, $\mathbb{S}q(Q(\mathbb{Z}_2[x]))$ is not complete. So $Q(\mathbb{Z}_2[x])$ is an example of an infinite field F such that $\mathbb{S}q(F)$ is not connected.

The following result considers the connectedness of $\mathbb{S}q(\mathbb{Q})$.

Theorem 5.19. $\mathbb{S}q(\mathbb{Q})$ *is connected.*

Proof. We have seen that $\mathbb{S}q(\mathbb{Z})$ is connected (cf. Theorem 5.1). So in $\mathbb{S}q(\mathbb{Q})$, there is a path between any two integer vertices. Now we show that there is a walk between any two square vertices, i.e., vertices of the form $\frac{a^2}{b^2}$, in $\mathbb{S}q(\mathbb{Q})$ (where $a, b \in \mathbb{Z}-\{0\}$). Consider two such vertices $\frac{x^2}{y^2}$ and $\frac{z^2}{t^2}$. Now $\frac{x^2}{y^2}+2x+y^2 = (\frac{x}{y}+y)^2$. Clearly, we can take x and z to be positive. Hence, $2x+y^2 \neq 0$ and $\frac{x}{y}+y \neq 0$. Thus, $\frac{x^2}{y^2} \leftrightarrow 2x+y^2$. Similarly $\frac{z^2}{t^2} \leftrightarrow 2z+t^2$. Now let P be a path from the integer vertex $2x+y^2$ to the integer vertex $2z+t^2$. So we have a walk $\frac{x^2}{y^2} \leftrightarrow 2x+y^2 \leftrightarrow P \leftrightarrow 2z+t^2 \leftrightarrow \frac{z^2}{t^2}$. Hence, there is a walk between any two vertices of the form $\frac{a^2}{b^2}$. Now consider any two vertices $\frac{a}{b}$ and $\frac{c}{d}$ (both distinct from 1 and 2). Then we have a walk $\frac{a}{b} \leftrightarrow 1-\frac{a}{b} \leftrightarrow \frac{a^2}{4b^2} \leftrightarrow P' \leftrightarrow \frac{c^2}{4d^2} \leftrightarrow 1-\frac{c}{d} \leftrightarrow \frac{c}{d}$, where P' is any walk between the square vertices $\frac{a^2}{4b^2}$ and $\frac{c^2}{4d^2}$. This shows that there is a path between any two vertices distinct from 1 and 2 in $\mathbb{S}q(\mathbb{Q})$. By Theorem 5.1, we already know that there is a path from 1 to -1 as well as a path from 2 to -1. So there is a walk between any two vertices in $\mathbb{S}q(\mathbb{Q})$. In other words, $\mathbb{S}q(\mathbb{Q})$ is connected. ∎

Now we look at infinite fields in general.

Theorem 5.20. *If F is an infinite field of characteristic 0, then $\mathbb{S}q(F)$ is connected.*

Proof. Let F be an infinite field and $Char(F) = 0$. So $\mathbb{Q}$ is a subfield of F. Let x be a non-zero element in F. We choose a non-zero rational number b such that $b \neq -\frac{1}{2x^2}, -\frac{1}{x^2}$. Note that $x^2, b^2 \neq 0$ since $x, b \neq 0$ and F is a field. Now $(x+\frac{1}{bx})^2 = x^2 + \frac{2}{b} + \frac{1}{b^2x^2}$ (Clearly, $x + \frac{1}{bx} \neq 0$ as $b \neq -\frac{1}{x^2}$). Now this gives that $x^2 \leftrightarrow \frac{2}{b} + \frac{1}{b^2x^2} \leftrightarrow -\frac{2}{b}$ (again, $\frac{2}{b} + \frac{1}{b^2x^2} \neq 0$, since $b \neq -\frac{1}{2x^2}$). Clearly, $\frac{2}{b}$ is a rational number. Hence, for any square element x^2, there is a walk from x^2 to some rational vertex. Since $\mathbb{S}q(\mathbb{Q})$ is connected, there is a path between any two vertices corresponding to rational numbers. So, all square element vertices lie in the same component in $\mathbb{S}q(F)$ (along with all rationals). Now let a be any non-zero non-square non-rational element in F. Then we have the walk $a \leftrightarrow 1-a \leftrightarrow \frac{a^2}{4}$ (note that since $a \neq 2$, i.e., $1-\frac{a}{2} \neq 0$, we have that $1-a+\frac{a^2}{4}$ is a square element). So for any non-square non-rational vertex a, there is walk from a to a square element vertex. Since we have already shown that all square element vertices and rational vertices are in a single component in $\mathbb{S}q(F)$, we have that $\mathbb{S}q(F)$ is connected. ∎

Remark 5.21. We know that $\mathbb{Z} \subset \mathbb{Z}[x] \subset Q(\mathbb{Z}[x])$, where $Q(\mathbb{Z}[x])$ is the quotient field of $\mathbb{Z}[x]$. Interestingly, $\mathbb{S}q(\mathbb{Z})$ is connected (by Theorem 5.1) and $\mathbb{S}q(Q(\mathbb{Z}[x]))$ is also connected by Theorem 5.20. However, as we have seen in Proposition 5.4, $\mathbb{S}q(\mathbb{Z}[x])$ is not connected.

Theorem 5.22. *If F is an infinite field such that $Char(F) = p$ for some odd prime p, then $\mathbb{S}q(F)$ is connected.*

Proof. Let F be an infinite field with $Char(F) = p$. So F has a subfield isomorphic to $\mathbb{Z}_p$. Consider any square element x^2 in F. By our definition of a square element, $x \neq 0$. Since $p > 2$, we have that 2 is a unit in F. First, let $x^2 \neq -2^{-2}, -2^{-1}$. Now $(x+2^{-1}x^{-1})^2 = x^2+1+2^{-2}x^{-2}$. Clearly, $x+2^{-1}x^{-1} = 0$ if and only if $x^2 = -2^{-1}$. So $x+2^{-1}x^{-1} \neq 0$ here, i.e., $x^2+1+2^{-2}x^{-2}$ is a square element. Now since $x^2 \neq -2^{-2}$, we have that $1 + 2^{-2}x^{-2} \neq 0$. Hence we have a walk $x^2 \leftrightarrow 1 + 2^{-2}x^{-2} \leftrightarrow -1$. So for any $x^2 \neq -2^{-2}, -2^{-1}$, there is a walk from x^2 to the vertex -1. So all these squares and (-1) lie in the same component (say, C) in $\mathbb{S}q(F)$. Now let $x^2 = -2^{-2}$. If $p = 3$, then $-2^{-2} = -1$ itself, and hence, lies in the component C. If $p = 5$, then $-2^{-2} = 1$. Now since F has infinitely many square elements (from Proposition 5.12), we can choose a square element $a^2 \neq 1, -2^{-1}, 2^{-1}, 4^{-1}$. So we have a walk $x^2(= 1) \leftrightarrow a^2 - 1 \leftrightarrow 4^{-1}a^{-2}$, which is a walk from x^2 to a distinct square element. Note that $a^2 - 1 \neq 0$, $a - 2^{-1}a^{-1} \neq 0$ and $4^{-1}a^{-2} \neq 1, -2^{-1}$ due to our choice of a^2. Hence, x^2 lies in the same component C. Again, if $p > 5$, then the subgraph induced by the subfield isomorphic to $\mathbb{Z}_p$ is connected. So there is a path from $x^2(= -2^{-2})$, which is a vertex of that subgraph, to another distinct square vertex in the same subgraph. So -2^{-2} lies in C. Now, let $x^2 = -2^{-1}$. For $p = 3, 5$, it is equal to 1 and 2 respectively, and hence, as before we can show that x^2 is in the component C (note that $2 \leftrightarrow -1$). For $p > 5$, there is a path from x^2 to some other distinct square vertex since the subgraph induced by the subfield isomorphic to $\mathbb{Z}_p$ is connected. So, we have shown that all square vertices in $\mathbb{S}q(F)$ lie in the same component. Now let b be a non-square element in $F - \{0\}$. If $b = 2$, then $b \leftrightarrow -1$ and hence, b is in the component C. If $b \neq 2$, then we have a walk $b \leftrightarrow 1 - b \leftrightarrow 4^{-1}b^2$ (note that $1 - b + 4^{-1}b^2$ is a square element, i.e., square of a non-zero element, if and only if $b \neq 2$). So there is a walk from b to the square element $4^{-1}b^2$, and consequently, b is in C. So all vertices in $\mathbb{S}q(F)$ are in the same component, i.e., $\mathbb{S}q(F)$ is connected. ■

Remark 5.23. The characteristic of a field is either 0 or a prime integer. From Theorem 5.20 and Theorem 5.22, we deduce that if F is an infinite field whose characteristic is not equal to 2, then $\mathbb{S}q(F)$ is connected. If the characteristic is 2, then $\mathbb{S}q(F)$ is not always connected, as we have seen in Remark 5.18.

Theorem 5.24. *If F is a field such that $Char(F) \neq 2$ and $F \ncong \mathbb{Z}_3, \mathbb{Z}_5$, then $\mathbb{S}q(F)$ is connected.*

Proof. This follows from Theorem 3.5 and Remark 5.23. ∎

Proposition 5.25. *Let $F_1, F_2, \ldots, F_k$ be fields such that for each $i = 1, 2, \ldots, k$ $(k > 2)$, either $F_i \cong \mathbb{Z}_3$ or $\mathbb{Z}_5$ or $\mathbb{S}q(F_i)$ is connected. Then $\mathbb{S}q(F_1 \times F_2 \times \cdots \times F_k)$ is connected.*

Proof. This can be easily proved by suitably combining Corollary 4.2, Theorem 4.4 and Corollary 4.6. ∎

Corollary 5.26. *If R is a commutative semisimple ring such that R has no element of additive order 2, then $\mathbb{S}q(R)$ is connected.*

Proof. If R is finite then the result follows from Corollary 4.8. So let R be an infinite semisimple commutative ring. Then R is a direct product of finitely many fields. Suppose that $R \cong F_1 \times F_2 \times \cdots \times F_k$, where the F_i's are fields. Now since there exists no element of additive order 2 in R, we have that $Char(F_i) \neq 2$ for all $i = 1, 2, \ldots, k$. Hence, from Theorem 5.24 and Proposition 5.25, the result follows. ∎

We now give another class of rings for which $\mathbb{S}q(R)$ is disconnected.

Proposition 5.27. *If F is a field of characteristic 2 such that $\mathbb{S}q(F)$ is not connected, then $\mathbb{S}q(F \times R)$ is not connected for any ring R.*

Proof. In $\mathbb{S}q(F \times R)$, a vertex (x, a) is not adjacent to a vertex of the form $(0, b)$ if x is a non-square in F. So if $\mathbb{S}q(F \times R)$ is connected, it is easy to see that we must have an adjacent pair of vertices (x, a) and (y, b), such that x is a square element in F and y is non-square in F. Let $x = t^2$. Now since (x, a) and (y, b) are adjacent, we have that either $x + y = 0$ or $x + y$ is a square element in F. Now $x + y = 0$ implies that $x = y$ which is impossible since x is a square whereas y is not. So $x + y = m^2$ for some $m \in F - \{0\}$. Now $x + y = m^2$ implies that $y = m^2 - x = m^2 - t^2 = m^2 + t^2 = (m + t)^2$, which contradicts that y is a non-square in F. So two such vertices cannot be adjacent. Hence, it follows that $\mathbb{S}q(F \times R)$ is not connected. ∎

Since there indeed exists a field $F = Q(\mathbb{Z}_2[x])$ of characteristic 2 such that $\mathbb{S}q(F)$ is disconnected, Proposition 5.27 gives us infinitely many rings for which $\mathbb{S}q(R)$ is disconnected. We now state the following result which can be proved without much difficulty from the definition of $\mathbb{S}q(R)$.

Theorem 5.28. *If $R_1 \cong R_2$, then $\mathbb{S}q(R_1) \cong \mathbb{S}q(R_2)$.*

However, the converse of the statement is not true, as we have seen in Remark 2.4. So we conclude the paper with the following question:

Problem 5.29. *Characterize the rings R_1, R_2 for which the following property holds:* $\mathbb{S}q(R_1) \cong \mathbb{S}q(R_2) \Leftrightarrow R_1 \cong R_2$.

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