B.A./B.Sc. 4th Semester (General) Examination, 2019

Subject : Mathematics

Paper : BMG4SEC21

(Vector Calculus)

Time: 2 Hours **Full Marks: 40**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Symbols and notation have their usual meaning.]

Group A

(Marks-10)

1. Answer *any five* questions: 2×5=10

(a) If $\vec{r} = (a\cos t)\hat{\imath} + (a\sin t)\hat{\jmath} + t\hat{k}$, then find $\left|\frac{d^2\vec{r}}{dt^2}\right|$.

(b) If $\vec{r} = e^{xy}\hat{\imath} + (2x-y)\hat{\jmath} + y\sin x\hat{k}$, then find $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}$.

(c) If $u = x^3 + 3yz^2$, then find $\vec{\nabla}u$, where $\vec{\nabla} \equiv$ grad.

(d) Find curl $\vec{v}$, where $\vec{v} = e^{xyz}(\hat{\imath} + \hat{\jmath} + \hat{k})$.

(e) If $\hat{e}$ be the unit vector, then show that div $\left((\hat{e}.\vec{r})\hat{e}\right) = 1$.

(f) If the vectors $\vec{A}$ and $\vec{B}$ are irrotational, then show that the vector $\vec{A} \times \vec{B}$ is solenoidal.

(g) If $\frac{d^2\vec{r}}{dt^2} = \vec{r}$, then show that $\vec{r} \times \frac{d\vec{r}}{dt}$ is a constant vector.

(h) Prove that if $\vec{r} = \vec{f}(t)$ has constant magnitude, then $\vec{f}.\frac{d\vec{f}}{dt} = 0$.

Group B

(Marks-10)

2. Answer *any two* questions: 5×2=10

(a) If $\vec{r} = (2x^2y - x^4)\hat{\imath} + (e^{xy} - y\sin x)\hat{\jmath} + (x^2\cos y)\hat{k}$, then find $\frac{\partial^2\vec{r}}{\partial x^2} \times \frac{\partial^2\vec{r}}{\partial y^2}$.

(b) Find the directional derivative of $\Phi = xy^2z + 4x^2z$ at (–1, 1, 2) in the direction of $(2\hat{\imath} + \hat{\jmath} - 2\hat{k})$.

(c) Let u be a scalar point function and $\vec{v}$ be a vector point function. If both u and $\vec{v}$ are differentiable, then show that div $(u\vec{v})$= grad $u.\vec{v} + u$ div $\vec{v}$.

(d) If $\vec{a}$ be a constant vector and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, then show that curl $\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = \frac{-\vec{a}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{a}.\vec{r})$.

Group C

(Marks-20)

3. Answer *any two* questions: 10×2=20

(a) (i) Show that the vector $(y\sin z - \sin x)\hat{\imath} + (x\sin z + 2yz)\hat{\jmath} + (xy\cos z + y^2)\hat{k}$ is irrotational.

(ii) If $\vec{F}$ be a solenoidal vector, then show that curl curl curl curl $\vec{F} = \vec{\nabla}^2\vec{\nabla}^2\vec{F} = \vec{\nabla}^4\vec{F}$.

(iii) If $f(x,y,z) = x^3 - y^3 + xz^2$, find grad f at (1, 1, –2). 4+4+2=10

(b) (i) If $f(x,y,z)$ is a scalar point function, then show that curl grad $f = \vec{0}$.

(ii) If $\vec{a}$ and $\vec{b}$ are constant vectors, then show that

grad $\{(\vec{r}\times\vec{a}).(\vec{r}\times\vec{b})\} = \vec{b}\times(\vec{r}\times\vec{a}) + \vec{a}\times(\vec{r}\times\vec{b})$, where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

(iii) If $r^2 = x^2 + y^2 + z^2$, then show that $\vec{\nabla}^2(ln\vec{r}) = \frac{1}{r^2}$. 3+4+3=10

(c) (i) If $\vec{r} = a\cos t\,\vec{\imath} + a\sin t\,\hat{\jmath} + bt\,\hat{k}$, then show that $\left[\frac{d\vec{r}}{dt}\,\frac{d^2\vec{r}}{dt^2}\,\frac{d^3\vec{r}}{dt^3}\right] = a^2b$.

(ii) If $\vec{F} = xy\,\hat{\imath} - 2xy^2\,\hat{\jmath} + zxy^3\,\hat{k}$ and $\vec{G} = 2x\,\hat{\imath} + y\,\hat{\jmath} - x^2z\,\hat{k}$, then find $\frac{\partial^2}{\partial x\partial y}(\vec{F}\times\vec{G})$ at (1, 1, –1). 5+5=10

(d) (i) Show that the vector $\sin y\,\hat{\imath} + \sin x\,\hat{\jmath} + e^z\hat{k}$ is neither solenoidal nor irrotational.

(ii) If f_1, f_2 are two arbitrary scalar point functions of x, y, z, then show that div (grad $f_1 \times$ grad f_2) = 0.

(iii) Show that, if $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $r = |\vec{r}|$, then curl $\left(\frac{\vec{r}}{r}\right) = \vec{0}$. 4+3+3=10

B.A./B.Sc. 4th Semester (General) Examination, 2019

Subject : Mathematics

Paper : BMG4SEC22

(Theory of Equations)

Time: 2 Hours **Full Marks: 40**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Symbols and notation have their usual meaning.]

Group A

(Marks-10)

1. Answer *any five* questions: 2×5=10

(a) Find a relation between c and d in order that $2x^4 - 7x^3 + cx + d$ may be exactly divisible by $x - 3$.

(b) If $x^3 + 3px + q$ has a factor of the form $(x - \alpha)^2$, show that $q^2 + 4p^3 = 0$.

(c) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of $\sum \alpha^2\beta$.

(d) Is $(x + 1)^4 + (1 + x^4) = 0$ a reciprocal equation? Justify your answer.

(e) Let α and β be two roots of the equation $px^2 + qx + r = 0, p \neq 0$. If p, q, r are in A.P. and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$ then find $|\alpha - \beta|$.

(f) If a and b are two roots of the equation $x^2 - 9x + 20 = 0$, find the value of $a^2 + b^2 + ab$.

(g) If α, β, γ be the roots of the equation $x^3 - 3x + 1 = 0$, then find the equation whose roots are $\alpha^2 + \beta\gamma, \beta^2 + \gamma\alpha, \gamma^2 + \alpha\beta$.

(h) If m and n be prime to each other them show that the equation $x^m - 1 = 0$ and $x^n - 1 = 0$ has no common root except unity.

Group B

(Marks-10)

2. Answer *any two* questions: 5×2=10

(a) (i) Using Descarte's rule of signs, find the number of positive, negative and complex roots of the equation $x^4 + 12x - 5 = 0$.

(ii) Show that the polynomial $x^3 + px^2 + qx + r$ will be perfect a cube if $p^3 = 27r$ and $q^2 = 3pr$.

(b) (i) If the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has three equal roots, show that each of those root is equal to $\frac{(6c-ab)}{(3a^2-8b)}$.

(ii) If α be an imaginary root of the equation $x^n - 1 = 0$, where n is prime number, then show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)\ldots(1-\alpha^{n-1}) = n$. 3+2=5

(c) (i) If α, β, γ be the roots of the equation $x^3 + 2x^2 + 1 = 0$, find the equation whose roots are $\alpha + \frac{1}{\alpha}$, $\beta + \frac{1}{\beta}$, $\gamma + \frac{1}{\gamma}$.

(ii) Find the equation whose roots are the reciprocal of the roots of the equation $x^4 + 2x^2 + 4x - 1 = 0$. 3+2=5

(d) (i) Solve: $x^3 - 18x - 35 = 0$ by Cardan's method.

(ii) A binomial equation of the form $x^n - 1 = 0$ has no multiple root. Justify. 4+1=5

Group C

(Marks-20)

3. Answer *any two* questions: 10×2=20

(a) (i) Show that all the roots of $\frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} = x$ are real.

(ii) Find the condition that the roots of equation $x^3 + px^2 + qx + r = 0$ are in A.P.

(iii) Find an equation where roots are the roots of the equation $x^4 + 5x^3 - 6x^2 + 8x - 9 = 0$ with their signs changed. 4+4+2=10

(b) (i) If the equation $f(x) = 0$ has all its roots real, then show that the equation $ff'' - \{f'\}^2 = 0$ has all its roots imaginary.

(ii) Solve the equation $x^4 - 8x^3 + 28x^2 - 48x - 13 = 0$, given that $2 - \sqrt{5}$ is one of its roots.

(iii) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, then find the value of $\sum(\alpha - \beta)^2$. 4+3+3=10

(c) (i) Applying a suitable transformation, remove the second term of the equation $x^4 + 4x^3 + 7x^2 + 6x - 4 = 0$ and hence solve it.

(ii) If α, β, γ are the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$, then find $\alpha^3 + \beta^3 + \gamma^3$ and also find $\alpha^4 + \beta^4 + \gamma^4$. 5+5=10

(d) (i) Solve $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.

(ii) Solve the biquadratic equation $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$. 5+5=10

B.A./B.Sc. 4th Semester (General) Examination, 2019

Subject : Mathematics

Paper : BMG4SEC23

Time: 2 Hours **Full Marks: 40**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
[Symbols and notation have their usual meaning.]

Group A

(Marks-10)

1. Answer *any five* questions: 5×2=10

(a) Prove that the square of any odd integer is of the form $8k+1$, k is an integer.

(b) Find two integers u and v satisfying $54u + 24v = 30$.

(c) Give an example to show that $a^2 \equiv b^2 \ (mod\ n)$ need not imply that $a \equiv b \ (mod\ n)$.

(d) State Chinese remainder theorem.

(e) If n be an odd integer (positive), prove that $\phi(2n) = \phi(n)$.

(f) Define the Möbius μ-function on the positive integers.

(g) Find $\tau(180)$.

(h) State Möbius inversion formula.

Group B

(Marks-10)

2. Answer *any two* questions: 5×2=10

(a) (i) Prove that the product of any m consecutive integers is divisible by m.

(ii) If a is prime to b, prove that $a+b$ is prime to ab. 3+2=5

(b) (i) Prove that the functions τ and σ are both multiplicative functions.

(ii) Prove $[a] + \left[a + \frac{1}{2}\right] = [2a]$ for all real a. 3+2=5

(c) (i) Define Euler's phi function.

(ii) If p is a positive prime and n is any positive integer, then

$$\phi(1) + \phi(p) + \phi(p^2) + \cdots ... + \phi(p^{n-1}) + \phi(p^n) = p^n.$$ 2+3=5

(d) (i) Find the remainder when $1! + 2! + 3! + \cdots + 50!$ is divided by 15.

(ii) If $n > 1$, the sum of all positive integers less than n and prime to n is $\frac{1}{2}n\phi(n)$. 3+2=5

Group C

(Marks-20)

3. Answer *any two* questions: 10×2=20

(a) (i) State and prove Lame's Theorem.

(ii) Show that the number of the form $a(a^2 + 2)/3$ is an integer, where a is an integer greater than or equal to 1 2+4+4=10

(b) (i) Define linear congruence.

(ii) Prove that $ax \equiv b(mod\ m)$, where g.c.d. $(a, m) = 1$, has a unique solution.

(iii) Solve $863x \equiv 880(mod\ 2151)$. 2+4+4=10

(c) (i) Let p be a prime and p does not divide a. Then prove that $a^{p-1} \equiv 1\ (mod\ p)$.

(ii) Prove that the linear Diophantine equation $ax + by = c$ has a solution if and only if $d|c$.

(iii) If $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ is the prime factorization of $n > 1$, then prove that

$$\tau(n) = (k_1 + 1)(k_2 + 1) \ldots (k_r + 1) \text{ and } \sigma(n) = \frac{p_1^{k_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{k_2+1} - 1}{p_2 - 1} \ldots \frac{p_r^{k_r+1} - 1}{p_r - 1}.$$

3+3+4=10

(d) (i) Let a and b be integers, not both zero. Then a and b are prime to each other if and only if there exist integers u and v such that $1 = au + bv$.

(ii) Use the theory of congruence to prove that $712^{5n+3} + 5^{2n+3}$ for all $n \geq 1$.

(iii) Prove that the total number of positive divisors of a positive integer n is odd if and only if n is a perfect square. 3+3+4=10

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