B.A./B.Sc. 3rd Semester (Honours) Examination, 2018 (CBCS)

Subject : Mathematics

(Theory of Real Functions & Introduction to Metric Spaces)

Paper : BMH3CC05

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols refer the usual meaning.

1. Answer *any ten* of the following questions: 2×10=20

(a) Using Sandwich theorem prove that $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$.

(b) Evaluate the limit : $\underset{x\to 0}{\text{Lt}}\,(1+2x)^{1/x}$.

(c) If $f : \mathbb{R} \to \mathbb{R}$ be a continuous map and $k \in \mathbb{R}$, then show that $\{x : f(x) = k\}$ is closed.

(d) Find the value of a , for which $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ x + a, & x \notin \mathbb{Q} \end{cases}$ is continuous at any point $x \in \mathbb{R}$.

(e) Give an example of a function which is monotonic on $[a, b]$ but does not satisfy intermediate value property on $[a, b]$.

(f) Prove or disprove that the existence of a derivative is necessary in order to have an extrema of a function on $\mathbb{R}$ at a point.

(g) Using mean value theorem prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

(h) Examine the differentiability of $f(x) = \sin|x| - |x|$ at $x = 0$.

(i) If d_1 and d_2 are two metrics on a non-empty set X, then prove that $d_1 + d_2$ is also a metric on X.

(j) Find the closure of the set $\left\{(x, y): 0 < x < 1; x \in \mathbb{Q};\ y = \sin\frac{1}{x}\right\}$ w. r. to the usual metric on $\mathbb{R}^2$.

(k) Prove that between any two real roots of the equation $e^x \cos x + 1 = 0$ there is at least one real root of the equation $e^x \sin x + 1 = 0$.

(l) Show that the function $f(x) = \cos\frac{1}{x}, 0 < x < 1$ is not uniformly continuous on (0,1).

(m) If $f : [0, 1] \to [0, 1]$ be continuous, prove that $\exists\, c \in [0, 1]$ such that $f(c) = c$.

(n) Let (x, d) be a metric space and $x \in X$, prove that the intersection of all neighbourhoods of x is the singleton $\{x\}$.

(o) Show that every set in a discrete metric space is open.

2. Answer *any four* of the following questions: 5×4=20

(a) (i) Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(x+y) = f(x) + f(y), \forall\ x, y \in \mathbb{R}$. If $\lim_{x\to 0} f(x) = L$ exists, then prove that $L = 0$ and f has a limit at every point $c \in \mathbb{R}$.

(ii) Show that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist (using Cauchy's principle). 3+2=5

(b) Let $f : [a,b] \to \mathbb{R}$ be a continuous function, $a \neq b$. If for any $k \in \mathbb{R}$ with $f(a) < k < f(b)$, show that, there exists a point $c \in (a,b)$ such that $f(c) = k$. Hence show that $f([a,b])$ is a closed and bounded interval. 2+3=5

(c) (i) Let $f : [a,b] \to \mathbb{R}$ be a function thrice differentiable on $[a,b]$. If $f(a) = f(b) = 0$, $f'(a) = f'(b) = 0$; then prove that $f'''(c) = 0$ for some $c \in (a,b)$.

(ii) Let $f : [a,b] \to \mathbb{R}$ be such that f' exists and bounded on $[a,b]$, then show that f is uniformly continuous on $[a,b]$. 3+2=5

(d) (i) Show that $l_p\ (1 \le p < \infty) = \{\{\alpha_n\} : \sum_{i=1}^{\infty} |\alpha_i|^p < \infty,\ \alpha_i \in \mathbb{R} \text{ or } \mathbb{C}\}$ is a metric space with respect to the map $dp : l_p \times l_p \to \mathbb{R}$ by $dp\ (x,y) = (\sum_{n=1}^{\infty} |\alpha_n - \beta_n|^p)^{1/p}$, $x = \{\alpha_n\}$, $y = \{\beta_n\}$ in l_p.

(ii) Examine whether $\mathbb{R} \setminus \mathbb{Z}$ is open or not. 4+1=5

(e) A function f is defined on [0, 1] by $f(0) = 1$ and

$$f(x) = 0, \text{ if } x \text{ is irrational}$$
$$= \frac{1}{n}, \text{ if } x = \frac{m}{n} \text{ where } m, n \text{ are positive integers prime to each other.}$$

Prove that f is continuous at every irrational point in [0, 1] and discontinuous at every rational point in [0, 1]. 5

(f) Let (X, d) be a metric space having finitely many dense subsets. Show that the number of dense subsets of X is of the form 2^n, for some integer $n \ge 0$.

3. Answer *any two* questions from the following: 10×2=20

(a) (i) Using $\epsilon - \delta$ definition show that $\lim_{x\to 0} \frac{1}{x^2} \notin \mathbb{R}$.

(ii) Use Taylor's theorem to prove that $\cos x \ge 1 - \frac{x^2}{2}$ for $-\pi < x < \pi$.

(iii) Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that for each $x \in [a,b]$, there exists $y \in [a,b]$ such that $|f(y)| \le \frac{1}{2}|f(x)|$. Prove that there exists a point $c \in [a,b]$ such that $f(c) = 0$.

(iv) Let $f : [a,b] \to \mathbb{R}$ be a continuous function with $f(x) > 0, \forall\ x \in [a,b]$. Then show that there exists a number $k > 0$ such that $f(x) \ge k, \forall\ x \in [a,b]$. 2+3+3+2=10

(b) (i) Find the expansion of $f(x) = \log\ (1 + x), x \in \mathbb{R}$ into infinite series in x in (–1, 1]. Also discuss its convergence in (–1, 1].

(ii) Determine the attitude of a right circular cylinder of greatest possible volume that can be inscribed in a sphere of radius r.

(iii) Let $f : [0,2] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} -1, & x \in [0,1) \\ 1, & x \in [1,2] \end{cases}$. Prove or disprove that there exists a function $g : [0,2] \to \mathbb{R}$ such that $g'(x) = f(x), \forall\, x \in [0,2]$. (2+3)+3+2=10

(c) (i) If (X, d) be a metric space and $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}, \forall\, x,y \in X$, then show that d & d_1 are equivalent metrics on X.

(ii) Prove that the intersection of a finite number of open sets in a metric space (X,d) is open. Show by an example that this result is not true for infinite number of open sets.

(iii) Show that the set of points of discontinuities of $f(x) = \lim_{n\to\infty} \frac{\left(1+\sin\frac{\pi}{x}\right)^n - 1}{\left(1+\sin\frac{\pi}{x}\right)^n + 1}, x \in (0,1)$ is bounded and countable. 4+(2+1)+3=10

(d) (i) If P_1 and P_2 be the radii of curvature at the ends of two conjugate diameters of an ellipse. Prove that $P_1^{2/3} + P_2^{2/3} = \frac{a^2+b^2}{(ab)^{2/3}}$

a and b being the lengths of semi-major and semi-minor axes of the ellipse.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable such that $f(0) = 0$, and $f\left(\frac{1}{n}\right) = 0, \forall\, n \in \mathbb{N}$, show that $f''(0) = 0$. 5+5=10

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