B.A./B.Sc. 3rd Semester (Honours) Examination, 2018 (CBCS)

Subject : Mathematics

(Group Theory-I)

Paper : BMH3CC06

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Symbols and notation have their usual meaning

Group-A

1. Answer *any ten* questions: $2\times10=20$

(a) How many rotations and how many reflections are there in the dihedral group D_7 of the symmetries of a regular heptagon?

(b) Prove that the inverse of an element in a group is unique.

(c) Let $n \geq 3$ and j be integers and $1 < j < n$. What is the inverse of j in the group $\mathbb{Z}_n$? Justify your answer.

(d) What is the order of the element $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the group $SL(2, \mathbb{R})$?

(e) Give an example to show that the union of two subgroups of a group may not be a subgroup of the group.

(f) Find all the subgroups of the group $\mathbb{Z}_{17}$.

(g) If $\langle a \rangle$ is a cyclic group of order 6, find all the generators of $\langle a \rangle$.

(h) Suppose that a cyclic group G has exactly three subgroups: G, {e} and a subgroup of order 3. What is the order of G?

(i) Show that the symmetric group S_3 is non-Abelian.

(j) Express the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 7 & 5 & 9 & 6 & 8 & 3 & 4 \end{pmatrix}$ as a product of disjoint cycles.

(k) Find the order of the permutation (1 2 4) (3 5 7 8).

(l) State the Lagrange's theorem on finite groups.

(m) Find all the distinct left cosets of the cyclic subgroup $\langle 3 \rangle$ of the group $\mathbb{Z}$.

(n) What is the order of the element (5, 3) in the group $\mathbb{Z}_{30} \oplus \mathbb{Z}_{12}$?

(o) Is $\mathbb{Z}_3 \oplus \mathbb{Z}_9$ isomorphic to $\mathbb{Z}_{27}$? Justify your answer.

Group-B

2. Answer *any four* questions: 5×4=20

(a) If G is the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are integers modulo p, where p is a prime integer, such that $ab - bc \neq 0$, prove that G forms a group relative to matrix multiplication. Find the order of G? 4+1=5

(b) (i) Prove that the group SL(2, $\mathbb{R}$) of 2×2 matrices with determinant 1 is a normal subgroup of GL(2, $\mathbb{R}$) of 2×2 matrices with non-zero determinant.

(ii) Let $f: (G,*) \rightarrow (G', o)$ be a homomorphism. Prove that, $ker f$ is a normal subgroup of G. 3+2=5

(c) Define centraliser $C(a)$ of an element a in a group G. Prove that $C(a)$ is a subgroup of G. 2+3=5

(d) (i) For any integer $n \geq 2$, prove that A_n, the alternating group of degree n, has order $n!/2$.

(ii) Show that A_3 is a normal subgroup of S_3. 3+2=5

(e) (i) Let G and H be finite cyclic groups. If $G \oplus H$ is cyclic, prove that $|G|$ and $|H|$ are relatively prime.

(ii) Is the group $\mathbb{Z} \oplus \mathbb{Z}$ cyclic? Justify your answer. 3+2=5

(f) (i) Let G be a group and $Z(G)$ be the centre of G. If $G/Z(G)$ is cyclic, prove that G is Abelian.

(ii) Show that $\mathbb{Z}/\langle n \rangle \approx \mathbb{Z}_n, n$ being a positive integer. 3+2=5

Group-C

3. Answer *any two* questions: 10×2=20

(a) (i) Let $G = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$. Show that G is a group under usual matrix multiplication.

(ii) Let G be a group of even order. Prove that there exists an element $a \neq e$ in G such that $a^2 = e$. (e is the identity in G)

(iii) How many elements of order 5 are there in the group $\mathbb{Z}_{10}$? 4+4+2=10

(b) (i) Prove that the alternating group A_4 has no subgroup of order 6.

(ii) Prove that the group $U(8)$ of all positive integers less than 8 and relatively prime to 8 under multiplication modulo 8 is not a cyclic group.

(iii) List all the subgroups of the group $\mathbb{Z}_{15}$. 4+4+2=10

(c) (i) Find all homomorphic images of the quaternion group Q_8. Show that dihedral group D_4 and Q_8 are not isomorphic.

(ii) Justify: Suppose H is a normal subgroup of a group G. Then G/H is commutative if and only if H is commutative. (4+2)+(2+2)=10

(d) (i) Find all subgroups of order 3 in $\mathbb{Z}_9 \oplus \mathbb{Z}_3$.

(ii) Let G be a group of order p^n, where p is prime. Prove that G contains an element of order p.

(iii) How many elements of order 5 are there in the group $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$? 3+3+4=10