B.A/B.Sc 3rd Semester (Honours) Examination, 2020 (CBCS)
Subject: Mathematics
Course: BMH3CC05 (Theory of Real Functions & Introduction to Metric Spaces)

Time: 3 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: $6\times5 = 30$

(a) A function f is defined on [0,1] by [5]

$f(0)=1$ *and*

$$f(x)=\begin{cases}0, \text{ if } x \text{ is irrational}\\ \frac{1}{n}, \text{ if } x=\frac{m}{n} \text{ where } m,n \text{ are positive integers prime to each other.}\end{cases}$$

Prove that f is continuous at every irrational point in [0,1] and discontinuous at every rational point in [0,1].

(b) A function $f:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$, and [5]

$f(x+y)=f(x)+f(y), \forall\, x,y\in\mathbb{R}$.

Prove that f is a linear function.

(c) State and prove Darboux's theorem on derivative. [1+4]

(d) Use Taylor's Theorem to prove that $1-\frac{1}{2}x^2\leq \cos x \text{ for } -\pi<x<\pi$. [5]

(e) Find the radius of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at an end of the major axis. [5]

(f) Let X be a nonempty set and d_1, d_2 be two metrics on X. Prove that [5]

$d: X\times X\to\mathbb{R}$, defined by $d(x,y)=\sqrt{(d_1(x,y))^2+(d_2(x,y))^2}$, is a metric on X.

(g) Prove that every closed ball in a metric space is a closed set in that metric space. [5]

(h) Prove that the metric space $l^p,\ 1\leq p<\infty$ is separable. [5]

2. Answer any three questions: $3\times10 = 30$

(a) (i) Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=\begin{cases}x, \text{if } x \text{ is rational}\\ 0, \text{if } x \text{ is irrational.}\end{cases}$ [3]

Prove that $\lim_{x\to a} f(x)$ exists only if $a=0$.

(ii) Show that $\lim_{x\to0-} e^{\frac{1}{x}}=0$. [2]

(iii) Prove that every real valued continuous function defined on a closed and bounded interval $[a,b]$ is bounded. [5]

(b) (i) If ρ_1, ρ_2 are the radii of curvature at the extremities of any chord of the cardioid $r = a(1+\cos\theta)$, which passes through the pole, then prove that $\rho_1^{\,2} + \rho_2^{\,2} = \frac{16}{9}a^2$. [5]

(ii) Find the equation of the evolute of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. [5]

(c) (i) Prove that a nonempty subset G in a metric space is open if and only if it is a union of open balls. [3]

(ii) Let (X,d) be a metric space and $A \subset X$. Prove that $\overline{A}$, the closure of A, is the intersection of all closed sets in (X,d) each containing A. [3]

(iii) Let $\mathbb{C}$ be the set of all complex numbers. Define $d: \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ by [4]

$$d(z_1, z_2) = \begin{cases} 0, & \text{if } z_1 = z_2 \\ |z_1| + |z_2|, & \text{if} z_1 \neq z_2. \end{cases}$$

Prove that d is a metric on $\mathbb{C}$.

(d) (i) State Cauchy's MVT and give its geometrical significance. [1+2]

(ii) A twice differentiable real valued function f defined on the closed and bounded interval $[a,b]$ is such that $f(a) = f(b) = 0$ and $f(x_0) < 0$ where $a < x_0 < b$. Prove that there exists at least one point $c \in (a,b)$ for which $f''(c) > 0$. [3]

(iii) If $x \in [0,1]$ prove that $\left|\log(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\right| < \frac{1}{4}$. [4]

(e) (i) If f is a real valued continuous function defined on a closed and bounded interval $[a,b]$, prove that f is uniformly continuous on $[a,b]$. [5]

(ii) Show that $f(x) = x^2$ is uniformly continuous in $(0,1)$. [2]

(iii) Define a Lipschitz function. If $I \subset \mathbb{R}$ is an interval and $f: I \to \mathbb{R}$ is a Lipschitz function, then prove that f is uniformly continuous on I. [3]