B.A/B.Sc. 3rd Semester (Honours) Examination, 2021 (CBCS)
Subject: Mathematics
Course: BMH3CC05
(Theory of Real Functions & Introduction to Metric Spaces)

Time: 3 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: $6\times5 = 30$

(a) State Cauchy's criterion for the existence of limit of a real function and use it to prove that $\lim\limits_{x\to0}\cos\dfrac{1}{x}$ does not exist. [1+4]

(b) A function $f:\mathbb{R}\to\mathbb{R}$ is continuous and $f(x+y)=f(x)f(y)\ \forall\ x,y\in\mathbb{R}$. Prove that either $f(x)=0\ \forall\ x\in\mathbb{R}, or\ f(x)=a^x\ \forall\ x\in\mathbb{R}$, where a is some positive real number and $\mathbb{R}$ being the set of all real numbers. [5]

(c) (i) If $f(x)=\sin x$, prove that $\lim\limits_{h\to0}\theta=\dfrac{1}{\sqrt{3}}$, where θ is given by $f(h)=f(0)+hf'(\theta h),\ 0<\theta<1.$ [3]

(ii) If $x\in[-1,1]$, prove that $\left|\sin x-\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}\right)\right|<\dfrac{1}{7!}.$ [2]

(d) Expand $f(x)=(1+x)^m$, where m is any positive real number. [5]

(e) If ρ_1,ρ_2 be the radii of curvature at the ends of two conjugate diameters of an ellipse, prove that $\left(\rho_1^{\frac{2}{3}}+\rho_2^{\frac{2}{3}}\right)(ab)^{\frac{2}{3}}=a^2+b^2.$ [5]

(f) Show that (X,d) is a metric space where $X=\mathbb{R}^2$, [5]

$$d(x,y)=\begin{cases}|x_1-y_1|\ if\ x_2=y_2\\ |x_1|+|y_1|+|x_2-y_2|\ if\ x_2\neq y_2\end{cases}$$

$for\ x=(x_1,x_2),\ y=(y_1,y_2)\ in\ X.$

(g) Prove that every open set in the space of real numbers can be expressed as a countable union of disjoint open intervals. [5]

(h) (i) Let $U=\{(x,y)\in\mathbb{R}^2:\ \ x\notin\mathbb{Z}, y\notin\mathbb{Z}\}$ where $\mathbb{Z}$ denotes the set of all integers. Is U an open set in $\mathbb{R}^2$ with respect to usual metric? Justify your answer. [2]

(ii) Let (X,d) be a metric space and $A \subset X$. Show that the set $S = \{x \in X : d(x,A) = 0\}$ is a closed set in X. Identify S in terms of A. [2+1]

2. Answer any three questions: 10×3 = 30

(a) (i) Prove that a continuous function on a closed bounded interval is uniformly continuous. [4]

(ii) If $f:\mathbb{R}\to\mathbb{R}$ is continuous at $x=c$ and $f(c)\neq 0$, then prove that there exists a certain neighbourhood of c at every point of which $f(x)$ will have the same sign as that of $f(c)$. [3]

(iii) Let $f,g:[a,b]\to\mathbb{R}$ be continuous functions on $[a,b]$. Let $\phi:[a,b]\to\mathbb{R}$ be defined by $\phi(x)=\max\{f(x),g(x)\}, x\in[a,b]$. Prove that ϕ is continuous on $[a,b]$. [3]

(b) (i) Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$ for $0 < x < \frac{\pi}{2}$. [4]

(ii) If f' exists and is bounded on some interval I, then prove that f is uniformly continuous on I. [3]

(iii) If a function $f:\mathbb{R}\to\mathbb{R}$ satisfies $|f(x)-f(y)|\leq (x-y)^2 \ \forall\ x,y\in\mathbb{R}$, then prove that f is constant. [3]

(c) (i) State and prove the Intermediate Value Property for derivatives. [1+2]

(ii) If [1+2]

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, \textit{when}\ x\neq 0 \\ 0, \textit{when}\ x=0 \end{cases}$$

$$\textit{and}\ g(x)=x\ \forall\ x\in\mathbb{R},$$

show that $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ does not exist but $\lim_{x\to 0}\frac{f(x)}{g(x)}$ exists and is equal to $\frac{f'(0)}{g'(0)}$.

(iii) Show that the radius of curvature at a point of the curve $x=ae^{\theta}(\sin\theta-\cos\theta)$, $y=ae^{\theta}(\sin\theta+\cos\theta)$ is twice the distance of the tangent at that point from the origin. [4]

(d) (i) Show that in the space $(\mathbb{R},d_u)$ with usual metric d_u, $\bigcap_{n=1}^{\infty}F_n$ is not a singleton, where [3]

$F_n=\left[-3-\frac{1}{n},-3\right]\cup\left[3,3+\frac{1}{n}\right]\ \forall\ n\in\mathbb{N}$. Give reasons.

(ii) For any two real numbers x, y define $\sigma(x, y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$. Show that σ is a metric on the set of real numbers. [3]

(iii) Prove that the space $l_p, 1 \leq p < \infty$ is separable. [4]

(e) (i) Prove that a closed sphere in a metric space is a closed set. [3]

(ii) In a metric space, prove that the derived set A' of a set A is a closed set. Is $(A')' = A'$? Justify your answer. [2+1]

(iii) Let (Y, d_Y) be a subspace of a metric space (X, d). Prove that a subset G of Y is open in (Y, d_Y) if and only if there exists an open set H in (X, d) such that $G = H \cap Y$. [4]