B.A/B.Sc. 3rd Semester (Honours) Examination, 2021 (CBCS)
Subject: Mathematics
Course: BMH3CC06 (Honours)
(Group Theory-I)

Time: 3 Hours Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: $6\times5 = 30$

(a) Show that if a group G has finite number of subgroups, then G is finite. [5]

(b) Show by an example that if H is a normal subgroup of G and K is a normal subgroup of H then K may not be a normal subgroup of G. [5]

(c) If G has only one element x of order n, then show that x is in Z(G),where Z(G) is the center of the group G. [5]

(d) If x is an element of group G of finite order and k is any non zero integer then show that $0(x)=0(x^k)$ if and only if k and $0(x)$ are relatively prime where $0(x)$ denotes the order of x. [5]

(e) Find the all 3-cycles in S_4 [5]

(f) Let n,m$\geq$ 2. Find all the homomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}_m$. [5]

(g) Let G be an abelian group of odd order and $f: G \to G$ be defined by $f(x) = x^2$ for all G. Show that f is an isomorphism. [5]

(h) Find all homomorphism from $\mathbb{Q}$ to $\mathbb{Z}$. [5]

2. Answer any three questions: $10\times3 = 30$

(a) (i) Let k be an integer and $f_k : \mathbb{R}^* \to \mathbb{R}^*$ be the map $f(x) = x^k$ for all x in $\mathbb{R}^*$. Is the map f_k an isomorphism? Justify your answer. [5]

(ii) Let G be a group and $x \neq e$ be an element in G. Then prove that there exists a unique homomorphism $f: \mathbb{Z} \to G$ such that $f(1) = x$. and also prove that kerf=$d\,\mathbb{Z}$ for some $d \neq 0$ iff $0(x)=d$. [5]

(b) (i) Let $G := \{A \in GL(2,\mathbb{R}): A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\}$. Show that $\mathbb{C}^*$ is isomorphic to the group G. [5]

(ii) Let G=$\mathbb{R}^2$ and $H = \{(x,0) \in G: x \in \mathbb{R}\}$. Show that $\frac{G}{H}$ is isomorphic to $(\mathbb{R},+)$. [5]

(c) (i) Let n be a positive integer and N=<n> be the cyclic subgroup of the additive group $\mathbb{Z}$ of integers. Show that $0(\frac{\mathbb{Z}}{N})=n$. [5]

(ii) If p is the smallest prime factor of the order of finite group G, prove that any subgroup of index p is normal. [5]

(d) (i) Let G= S_3 and H=A_3. Show that G/H is isomorphic to {1,-1}. [5]

(ii) Let G be a group with at least two elements such that G has no subgroup other than $\{e\}$ and itself. Then prove that G is a cyclic group of prime order. [5]

(e) (i) Let G be a finite group. Then prove that every element in G is of finite order. Is the converse true? Justify your answer. [5]

(ii) Let G be an infinite group. Show that G has infinitely many proper subgroups. [5]