B.A./B.Sc 4th Semester (Honours) Examination, 2019 (CBCS)
Subject : Mathematics
Paper : BMH4 CC08

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

[Notations and Symbols have their usual meaning.]

Group-A
Marks : 20

1. Answer *any ten* questions: 2×10=20

(a) Evaluate : $\int_0^3 [x]\,dx$, where $[x]$ denotes the largest integer not larger than real number x.

(b) Let $f(x) = x$ if $x \neq \frac{1}{n}$

$f\left(\frac{1}{n}\right) = 1$ for $n = 1, 2, 3$ ....

Is $f(x)$ Riemann integrable in $[0, 1]$? Support your answer.

(c) Give examples of two functions f and g which are not Riemann integrable in [0, 1] but $\phi(x) = Max\{f(x), g(x)\}$ is Riemann integrable in [0, 1]. Answer with justification.

(d) Find $\lim_{x\to 0} \frac{1}{x} \int_0^x \cos\frac{1}{t^2}\,dt$, if it exists.

(e) Test the convergence of $\int_0^\infty e^{-x^2}\,dx$.

(f) Let f and g be two continuous functions defined over $[a, b]$, $a < b$, such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Show that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

(g) Test for convergence of the integral $\int_0^1 \frac{\sqrt{x}}{\text{Sin}x}\,dx$.

(h) With proper justification give an example of a sequence of functions $\{f_n\}$ which converges pointwise to a function f such that each f_n is Riemann integrable but f is not Riemann integrable.

(i) Test for uniform convergence of the sequence $\{(\sin x)^n\}$ in the interval [1, 2].

(j) Let $f_n(x) = 1$ if $\frac{1}{n} \leq x \leq 1$ and

$= nx$ if $0 \leq x < \frac{1}{n}$.

Find $\lim_{n\to\infty} f_n(x)$ in [0, 1], if it exists.

(k) Test the series $\sum_{n=1}^{\infty} x^n(1-x^n)$ for pointwise convergence and uniform convergence in [0, 1].

(l) Give an example of a power series which converges nowhere except $x = 0$. Justify your answer.

(m) Find the region of convergence of the power series $\sum_{n=1}^{\infty} \frac{n!(x+2)^n}{n^n}$.

(n) Examine whether $\sum_{n=1}^{\infty} (\sin nx + \cos nx)$ is a Fourier series for some bounded integrable function over $[-\pi, \pi]$.

(o) Show that the Fourier series of a bounded integrable odd function over $[-\pi, \pi]$ consists of sine terms only.

Group-B

Marks : 20

2. Answer *any four* questions from the following: 5×4=20

(a) State and prove Fundamental Theorem of Integral Calculus. 2+3=5

(b) If $\{f_n(x)\}$ is a sequence of continuous functions on $[a, b]$ and if $f_n(x) \to f(x)$ uniformly in $[a, b]$ as $n \to \infty$, then prove that $f(x)$ is continuous in $[a, b]$.

(c) (i) Prove that every monotone function over $[a, b]$ is Riemann integrable therein.

(ii) Give an example with proper justification of a function f on $[0, 1]$ which is not Riemann integrable on $[0, 1]$ but $|f|$ is Riemann integrable on $[0, 1]$. 3+2=5

(d) Suppose $f(x) = x^2$ in $0 \le x \le 2\pi$ and $f(x)$ is a periodic function of period $= 2\pi$. Obtain the Fourier series of f and examine the convergence of the series in $[0, 2\pi]$. 3+2=5

(e) State and prove Weierstrass *M*-test for uniform convergence of a series of functions. 2+3=5

(f) State Dirichlet's condition concerning convergence of Fourier series of a function. If f is bounded and integrable in $[-\pi, \pi]$ and if a_n, b_n are its Fourier co-efficients, then prove that $\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ is convergent. 2+3=5

Group-C

Marks : 20

3. Answer *any two* questions from the following: 10×2=20

(a) (i) Let f be a bounded function defined on the closed interval $[a, b]$. Prove that a necessary and sufficient condition that f be Riemann integrable over $[a, b]$ is that to every $\epsilon > 0$ there corresponds a $\delta > 0$ such that for every partition P of $[a, b]$ with $\|P\| \leq \delta$, the oscillatory sum $W(P, f) < \epsilon$.

(ii) If a bounded function f is Riemann integrable in $[a, b]$, show that $|f|$ is also Riemann integrable in $[a, b]$ and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx$.

(iii) Show that $\left|\int_a^b \frac{\sin x}{x} dx\right| \leq \frac{4}{a}$ for $0 < a < b < \infty$. 4+4+2=10

(b) (i) Prove the relation $\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, $m > 0,\ n > 0$.

(ii) Let $f(x) = |x|$ for $-\pi < x \leq \pi$ and $f(x + 2\pi) = f(x)$ for all x. Find the Fourier series of f. Hence deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$. 5+(4+1)=5

(c) (i) Test the convergence of the integral $\int_0^\infty e^{-x} x^{n-1} dx$, for different values of n.

(ii) Show that $\frac{1}{2}\{\log(1 + x)\}^2 = \frac{1}{2}x^2 - \frac{1}{3}\left(1 + \frac{1}{2}\right)x^3 + \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^4 - \cdots \ \forall\, x \in (-1, 1)$.

5+5=10

(d) (i) Suppose that a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a finite non-zero radius of convergence ρ. Prove that the series converges uniformly in each compact subset of $(-\rho, \rho)$.

(ii) If two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge to the same sum function in an interval $(-r, r)$, $r > 0$, then show that $a_n = b_n$ for all n.

(iii) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$, where $a_{2n} = \frac{1}{3^n}$, $a_{2n-1} = \frac{1}{3^n+1}$.

4+3+3=10

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