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B.A./B.Sc. 4th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH4CC09

(Multivariate Calculus)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

Group-A

(Marks : 20)

1. Answer *any ten* questions: 2×10=20

(a) Show that the function $f(x,y)$ defined by $f(x,y)=\begin{cases}\frac{xy}{\sqrt[3]{x^2+y^2}}, & \text{when } x^2+y^2\neq 0\\ 0, & \text{when } x^2+y^2=0\end{cases}$ is continuous at (0,0).

(b) Show that $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^2}{x^2y^2+(x-y)^2}$ does not exist.

(c) Find $\frac{\partial z}{\partial\theta}$ from the relation $z=\log\sin(x^2y^2-1), x=r\cos\theta, y=r\sin\theta$.

(d) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} y^2dydx$.

(e) Evaluate $\int_1^e\int_1^2 \frac{1}{xy}\,dydx$.

(f) Prove that $f(x,y)=|x|+|y|$ is not differentiable at $(0,0)$.

(g) Let $f(x,y)=\frac{x^2y}{x^4+y^2}$. Discuss the existence of the limit of $f(x,y)$ as $(x,y)\to(0,0)$.

(h) Prove that $\lim_{x\to 0}\lim_{y\to 0}\frac{x^2+y^2}{x^2-y^2}\neq\lim_{y\to 0}\lim_{x\to 0}\frac{x^2+y^2}{x^2-y^2}$.

(i) Let $f(x,y)$ be defined as

$$f(x,y)=\begin{cases}1, \text{if } xy\neq 0\\ 0, \text{if } xy=0\end{cases}$$

Prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(j) If $\vec{a}=(2y+3)\hat{\imath}+xz\hat{\jmath}+(yz-x)\hat{k}$, evaluate $\int_\Gamma \vec{a}.d\vec{r}$, where Γ is the curve $x=2t^2$, $y=t, z=t^3$ from $t=0$ to $t=1$.

(k) Find the constants a,b,c so that the vector $\vec{F}=(x+2y+az)\hat{\imath}+(bx-3y-z)\hat{\jmath}+(4x+cy+2z)\hat{k}$ is irrotational.

(l) Use Gauss's divergence theorem to show that $\iint_S \vec{r}.d\vec{s} = 3V$, where V is the volume enclosed by the closed surface S and $\vec{r}$ has its usual meaning.

(m) Show that grad f is a vector perpendicular to the surface $f(x,y,z) = c$, where c is a constant.

(n) If the vectors $\vec{A}$ and $\vec{B}$ are irrotational, then show that the vector $\vec{A} \times \vec{B}$ is solenoidal.

(o) Use Stoke's theorem to prove that $\int_c \vec{r}.d\vec{r} = 0$.

Group-B

(Marks : 20)

2. Answer *any four* questions: 5×4=20

(a) Show that $f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$, possesses partial derivatives at (0,0) but is not differentiable at (0,0).

(b) If $\frac{u}{x} = \frac{v}{y} = \frac{w}{z} = (1-r^2)^{-\frac{1}{2}}$ where $r^2 = x^2+y^2+z^2$, then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = (1-r^2)^{-\frac{5}{2}}$.

(c) Show that $\iiint e^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}} dxdydz$ taken throughout the region $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1$ is $4\pi abc(e-2)$.

(d) If $f(0) = 0, f'(x) = \frac{1}{1+x^2}$, prove without using the method of integration that $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$.

(e) Evaluate $\oint_C \vec{F}.d\vec{r}$ by Stoke's theorem where $\vec{F} = y^2\hat{\imath} + x^2\hat{\jmath} - (x+z)\hat{k}$ where C is the boundary of the triangle with vertices (0,0,0), (1,0,0), (0,1,0).

(f) Find the values of the constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum of magnitude 64 in a direction parallel to the z-axis.

Group-C

(Marks : 20)

3. Answer *any two* questions: 10×2=20

(a) (i) State and prove Young's theorem for commutativity of second order partial derivatives, of a function of two variables.

(ii) Give an example to show that the conditions of the theorem are not necessary. (1+4)+5

(b) (i) State Euler's theorem and its converse for a homogeneous function in x,y,z. Use it to prove that if $f(x,y,z)$ is a homogeneous function in x,y,z of degree n having continuous partial derivatives then $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are each homogeneous function in x,y,z of degree $n-1$.

(ii) If H is a homogeneous function in x, y, z of degree n and $u = (x^2+y^2+z^2)^{-\frac{1}{2}(n+1)}$, then prove that $\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(H\frac{\partial u}{\partial z}\right) = 0$. 5+5

(c) (i) Prove that for any two vector functions $\vec{f}$ and $\vec{g}$, $div\,(\vec{f}\times\vec{g}) = \vec{g}.curl\,\vec{f} - \vec{f}.curl\vec{g}$.

(ii) Prove that $\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}.\vec{F}) - \vec{\nabla}^2\vec{F}$.

(iii) Give the physical interpretation of divergence of a vector function. 4+4+2

(d) (i) For the function f defined as:

$$f(x,y) = \begin{cases} \frac{1}{y^2}, & \text{if } 0 < x < y < 1 \\ \frac{1}{x^2}, & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise if } 0 \le x, y \le 1, \end{cases}$$

show that $\int_0^1 dx \int_0^1 f\ dy \neq \int_0^1 dy \int_0^1 f\ dx$. Does the double integral $\iint_R f\,dx\,dy$ exist?

(ii) Find the value of $\iint_E e^{\frac{y}{x}}\ dS$ if the domain E of integration is the triangle bounded by the straight lines $y = x, y = 0$ and $x = 1$.

(iii) Using Green's theorem in the plane, evaluate $\oint_C (2x - y^3)dx - xydy$, where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. 4+3+3

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