B.A./B.Sc. 1st Semester (Honours) Examination, 2017 (CBCS)

Subject : Mathematics

Paper: BMHICC-II

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) If $z = (7 - 3\sqrt{3}i)(-1 - i)$, find $\arg(-z)$.

(b) If n be a positive integer, prove that $\left(1 + \frac{1}{1+n}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^{n}$.

(c) Show that the equation $x^3 + 2x^2 - 2x - 1 = 0$ has one positive root and two negative roots—one lying between–3 and –1 and another lying between –1 and 0.

(d) If α is a multiple root of order 3 of the equation $x^4 + bx^2 + cx + d = 0$ $(d \neq 0)$, show that $\alpha = \frac{8d}{3c}$.

(e) State De Moivre's theorem.

(f) Solve the equation $x^4 + x^2 - 2x + 6 = 0$, it is given that $1 + i$ is a root.

(g) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x}{\sqrt{x^2+1}}$. Show that f is one-one.

(h) Let $f: A \to B$ and $g: B \to C$ be two invertible functions. Then show that gof is invertible.

(i) Let $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a^2 - b^2$ is divisible by $3\}$. Is R an equivalence relation on $\mathbb{Z}$? Justify your answer.

(j) Prove that $n^2 - 2$ is never divisible by $4, n \in \mathbb{Z}$.

(k) Use Cayley-Hamilton theorem to find A^{-1}, where $A = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$.

(l) Prove that in a vector space two non-zero vectors are linearly dependent iff each vector is a scalar multiple of the other.

(m) Find a basis for the given subspace W of $\mathbb{R}^3$ where $W = \{(a, b, c) \in \mathbb{R}^3 : b = a + c\}$.

(n) Prove that the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$T(x, y, z) = (x + y,\ \ y + z,\ \ z + x)$ for $(x, y, z) \in \mathbb{R}^3$ is one-to-one.

(o) Does there exist a linear transformation T defined on the subspace of all 2×1 matrices over the field R to itself such $T\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$ and $T\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}$? Justify.

2. Answer *any four* questions: 5×4=20

(a) If $a, b, c, d > 0$ and $a + b + c + d = 1$, prove that $\frac{a}{1+b+c+d} + \frac{b}{1+a+c+d} + \frac{c}{1+a+b+d} + \frac{d}{1+a+b+c} \geq \frac{4}{7}$.

(b) Use Sturm's function to show that roots of the equation $x^3 + 3x^2 - 3 = 0$ are real and distinct.

(c) State and prove division algorithm theorem.

(d) Find a linear operator T on $\mathbb{R}^3$ such that $\ker T$ is the subspace $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ of $\mathbb{R}^3$.

(e) (i) If $f : A \to B$ and $g : B \to C$ be two mappings such that $gof : A \to C$ is injective, then prove that f is injective. Give an example, to show that gof is injective but g is not injective. 2

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x + 8,\ x \in \mathbb{R}$. Prove that f is invertible. Find f^{-1}. 3

(f) The matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$ and $\{(1, 0), (1, 1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix}1 & 2 & 4\\2 & 1 & 0\end{pmatrix}$. Find T. Also find the matrix of T relative to the ordered bases $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ of $\mathbb{R}^3$ and $\{(1, 1), (0, 1)\}$ of $\mathbb{R}^2$.

3. Answer *any two* questions: 10×2=20

(a) (i) Prove that $\cos^5\theta = \frac{1}{16}[\cos 5\theta + 5\cos 3\theta + 10\cos\theta]$. 5

(ii) Show that the equation $(x - a)^3 + (x - b)^3 + (x - c)^3 + (x - d)^3 = 0$ where a, b, c, d are positive and not all equal, has only one real root. 5

(b) (i) Find dim $(S \cap T)$, where S and T are subspaces of the vector space $\mathbb{R}^4$ given by

$S = \{(x, y, z, w) \in \mathbb{R}^4 : 2x + y + 3z + w = 0\}$,

$T = \{(x, y, z, w) : x + 2y + z + 3w = 0\}$. 4

(ii) Find the rank of the matrix: 3

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$$

(iii) Prove that the eigenvalues of a real symmetric matrix are all real. 3

(c) (i) Let $f : A \to B$ be a mapping. Show that f is onto iff there exists a mapping $g : B \to A$ such that $f \circ g = I_B$. 4

(ii) Prove that $1 + 2 + \dots\dots\dots + n$ is a divisor of $1^r + 2^r + \dots\dots\dots + n^r$ for any odd positive integer r. 3

(iii) If $p \geq q \geq 5$ and p, q are both primes, prove that $24|(p^2 - q^2)$. 3

(d) (i) Solve, if possible, the system of equations 5

$$x + 2y + z - 3w = 1$$
$$2x + 4y + 3z + w = 3$$
$$3x + 6y + 4z - 2w = 4$$

(ii) Determine the conditions for which the system of equations 5

$$x + 2y + z = 1$$
$$2x + y + 3z = b$$
$$x + ay + 3z = b + 1$$

has (I) only one solution, (II) no solution, (III) many solutions.
