B.A/B.Sc. 1st Semester (Honours) Examination, 2020 (CBCS)
Subject: Mathematics
Course: BMHCC02 (Algebra)

Time: 3 Hours Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: 6×5 = 30

(a) Solve the equation $x+y+z=2, x^2+y^2+z^2=22, x^3+y^3+z^3=8$. [5]

(b) If a, b, c are all positive real numbers and $a+b+c=1$ prove that [5]

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} \geq \frac{9}{2}$$

(c) (i) What is the remainder when $6.7^{32}+7.9^{45}$ is divided by 4? [3]

(ii) If $S_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, prove that $n+S_n > n\,(n+1)^{1/n}$, $if\ n>1$. [2]

(d) Find the standard matrix A for the transformation $T(X)=3X$ for all X in $\mathbb{R}^2$. [5]

(e) Prove that any integer of the form $6k+5$ is also of the form $3r-1$ for some r in $\mathbb{Z}$ but not conversely. [5]

(f) If α, β, γ be the roots of the equation $x^3+qx+r=0$ $(r \neq 0)$, find the equation whose roots are $\frac{\beta}{\gamma}+\frac{\gamma}{\beta}, \frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}, \frac{\alpha}{\beta}+\frac{\beta}{\alpha}$. [5]

(g) Suppose A is 2×2 real matrix with trace 5 and determinant 6. Find the eigenvalues of the matrix B, where $B=A^2-2A+I$ [5]

(h) Let us consider the function $f:\mathbb{R}\to(-1,1)$ given by $f(x)=\frac{x}{1+|x|}$ for all x in $\mathbb{R}$. Show that f is a bijective function and find f^{-1}. [5]

2. Answer any three questions: 10×3 = 30

(a) (i) State Caley-Hamilton theorem. Use this theorem to find A^{100}, where. [1+4]

$$A=\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

(ii) Show that the product of all values of $(1+\sqrt{3}\,i)^{3/4}$ is 8. [5]

(b) (i) Prove that any non-empty open interval in real line is uncountable. [5]

(ii) If x, y, z are positive rational numbers, prove that [5]

$$\left(\frac{x^2+y^2+z^2}{x+y+z}\right)^{x+y+z} \geq x^x y^y z^z \geq \left(\frac{x+y+z}{3}\right)^{x+y+z}.$$

(c) (i) Let V be the set of all points (x, y) in the xy plane. Define a relation R given by $(x,y)\ R\ (a,b)$ if and only if $x^2+y^2=a^2+b^2$. Show that R is an equivalence [2+2+1] relation on V. Determine the equivalence classes of R? Determine the equivalence class containing the point $(2,3)$.

(ii) Show that there does not exist any natural number n so that dimension of the subspace of $M_{n\times n}$, consisting of $n \times n$ matrices whose sum of elements in each row and diagonals are equal to 0, is $n^2 + n$. [5]

(d) (i) Determine h and k so that the solution set of system of equations $x_1 + y_1 = k$ and $4x_1 + hy_1 = 5$. [5]

a) is empty

b) is a singleton

c) is infinite.

(ii) Show that there does not exist any surjective map from an arbitrary set X onto $\mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the powerset of X. [5]

(e) (i) Use Sturm's theorem to show that the roots of the following equation are all real and distinct: $x^4 + 4x^3 - x^2 - 10x + 3 = 0$. [5]

(ii) If z is a non-zero complex number and m, n are positive integers prime to each other, then show that $\left(z^{1/n}\right)^m = (z^m)^{1/n}$. [5]