B.A/B.Sc.1st Semester (Honours) Examination, 2021 (CBCS)
Subject: Mathematics
Course: BMH1CC02 (Algebra)

Time: 3 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: $6\times5 = 30$

(a) Show that there does not exist any surjective map from a set X to its power set $\mathcal{P}(X)$. [5]

(b) If α, β, γ are the roots of the equation

$$x^3 - 7x^2 + x - 5 = 0,$$

find the equation whose roots are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$. [5]

(c) Show that for all odd integers n, $gcd(3n,3n+2)=1$. [5]

(d) Calculate the Sturm's function and locate the position of the real roots of the equation $x^3 - 3x - 1 = 0$. [5]

(e) Show that $\{(1, \alpha, \alpha^2), (1, \beta, \beta^2), (1, \gamma, \gamma^2)\}$ is a linearly independent subset of $\mathbb{R}^3$ for all distinct real numbers α, β, γ. [5]

(f) State and prove De Moivre's theorem for integral values of index. [1+4]

(g) Show that the eigenvalues of a real symmetric matrix are real. Is the converse true? Justify your answer. [3+2]

(h) If p is prime and p divides ab, where a and b are integers. Prove that either p divides a or p divides b. [5]

2. Answer any three questions: $10\times3 = 30$

(a) (i) For any eigenvalue α of matrix $A = (a_{i,j})_{n\times n,}$ show that the set of all eigenvectors of A belonging to α is a subspace of $\mathbb{R}^n$. [4]

(ii) State Caley-Hamilton theorem. Using it, determine the inverse of the matrix, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. [1+5]

(b) (i) Let X be an eigenvector of an $n \times n$ real matrix A associated with an eigenvalue α. Prove that $P^{-1}X$ is an eigenvector of the matrix $P^{-1}AP$ associated with α. [4]

(ii) If d= gcd(a,b), then prove that there exists $u, v \in \mathbb{Z}$ such that $d= au+bv$ where a, b are two non zero integers. [6]

(c) (i) For $a, b \in \mathbb{N}$, define $a \backsim b$ if and only if $a^2 + b$ is even. Prove that $\sim$ defines an equivalence relation. [5]

(ii) If a, b, c are positive real numbers, prove that $a^4 + b^4 + c^4 \geq abc(a + b + c)$. [5]

(d) (i) If $2cos\theta = x + \frac{1}{x}$ and θ is real, prove that $2\cos n\theta = x^n + \frac{1}{x^n}$, n being an integer. [5]

(ii) Prove that a mapping $f: X \to Y$ is one-to- one iff $f(A \cup B) = f(A) \cup f(B)$. [5]

(e) (i) Let $X = \{1,2,3\}$. Find all equivalence relations on X. [6]

(ii) Show that the number of primes is infinite. [4]