**B.A./B.Sc. 6th Semester (Honours) Examination, 2021 (CBCS)**
**Subject: Mathematics**
**Course: BMH6CC13**
**(Metric spaces and Complex Analysis)**

Time: 3 Hours | Full Marks: 60

*The figures in the margin indicate full marks.*
*Candidates are required to write their answers in their own words as far as practicable.*
[Notation and Symbols have their usual meaning]

**1. Answer any six questions:** $6\times5 = 30$

(a) Let $(X,d)$ be a metric space. Prove that any two disjoint closed sets in $(X,d)$ can be separated by disjoint open sets in $(X,d)$. [5]

(b) Let $X = C[0,1]$, the set of all real valued continuous functions defined over the closed interval $[0,1]$, and let $d(f,g) = \int_0^1 |f(t) - g(t)|\,dt,\ f,g \in X$.
Prove that $(X,d)$ is not complete. [5]

(c) Define a Lebesgue number with respect to an open cover in a metric space. Prove that in a sequentially compact metric space every open cover has a Lebesgue number. [1+4]

(d) Show that the unit sphere $S = \left\{ x = \{x_n\} \in l_2 :\ \sum_{n=1}^{\infty} x_n^{\,2} \le 1 \right\}$ is not compact. [5]

(e) If $u - v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x+iy$, find $f(z)$ in terms of $z$. [5]

(f) If $f$ is an analytic function, prove that $\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2$. [5]

(g) If $f$ is an analytic function on a positively oriented simple closed rectifiable contour $C$ and on *Int*(C), then prove that $f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$ for any $z_0 \in Int(C)$ and for $n = 0,1,2,\ldots$ [5]

(h) (i) Obtain the Laurent series representation [3]

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{(z-1)^n},\ \text{when } 1 < |z-1| < \infty.$$

(ii) If $C$ is a closed contour around the origin, prove that $\left( \frac{a^n}{n!} \right)^2 = \frac{1}{2\pi i} \int_C \frac{a^n e^{az}}{n!\, z^{n+1}}\, dz.$ [2]

**2. Answer any three questions:** 10×3 = 30

(a) (i) Prove that the image of a Cauchy sequence under a uniformly continuous function is Cauchy. Is the result true if *f* is continuous? Support your answer. [2+2]

(ii) Prove that in a metric space if a connected set is contained in the union of two separated sets then it is contained in exactly one of them. [3]

(iii) Let $f : X \to \mathbb{R}$ be a non-constant continuous function, where $(X, d)$ is connected. Prove that $f(X)$ is uncountable. [3]

(b) (i) Prove that the space $C[0,1]$ of all real valued continuous functions on $[0,1]$ is complete but not compact with respect to the sup metric on $C[0,1]$. [4+3]

(ii) Let $(X, d)$ and $(Y, \rho)$ be two metric spaces and $f : X \to Y$ be a continuous function. If $X$ is compact, prove that *f* is a closed mapping. [3]

(c) (i) State and prove Cauchy-Hadamard theorem on power series of complex numbers. [1+6]

(ii) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n\sqrt{2}+i}{1+2ni} z^n$. [3]

(d) (i) If $f(z) = u + iv$ is differentiable at $z_0 = x_0 + iy_0$, prove that $u_x, u_y, v_x, v_y$ exist and $u_x = v_y, u_y = -v_x$ at $(x_0, y_0)$. [4]

(ii) Show that a real function of a complex variable either has derivative zero or derivative does not exist. [3]

(iii) Let $f(z)$ be analytic in a non-empty connected open set $D \subset \mathbb{C}$ and $f^{/}(z) = 0 \; \forall \; z \in \mathrm{D}$. Prove that *f* is constant on *D*.

(e) (i) Let *C* be a closed contour of length *L* and $f(z)$ is a piecewise continuous function on *C*. If *M* is a non-negative constant such that $|f(z)| \leq M \; \forall \; z \in C$, then prove that $\left| \int_C f(z)\, dz \right| \leq M \cdot L.$ [3]

(ii) Using Liouville's theorem, prove the fundamental theorem of algebra. [4]

(iii) Without evaluating the integral, show that $\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3}$, where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. [3]