B.A./B.Sc. 6th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH6CC-XIV

(Ring Theory and Linear Algebra-II)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Show that if $a \in R$ is such that Ra is a maximal ideal, then a is an irreducible element, where R is commutative ring with identity.

(b) Prove that every Euclidean domain has a unit element.

(c) Let $f(x) \in \mathbb{Z}_p[x]$, p being a prime. Prove that if $f(b) = 0$, then $f(b^p) = 0$.

(d) Show that every field is an Euclidean domain.

(e) Let F be an infinite field and let $f(x) \in F[x]$. If $f(a) = 0$ for infinitely many elements a of F; then show that $f(x) = 0$, i.e. $f(x)$ is a null polynomial.

(f) Prove that any two elements of a PID have a gcd.

(g) Show that in a PID every non-zero prime ideal is a maximal ideal.

(h) Prove that any orthogonal set of non-zero vectors in an inner product space is linearly independent.

(i) Let T be a linear operator on an inner product space V and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

(j) Let $S = \{(1, 0, i), (1, 2, 1)\}$ in $\mathbb{C}^3(\mathbb{C})$. Compute the orthogonal complement $S^\perp$ of S.

(k) Consider the real inner product space $\mathbb{R}^3$. Find the orthogonal projection of the given vector u on the subspace W of $\mathbb{R}^3$, where $u = (2, 1, 3)$, $W = \{(x, y, z): x + 3y - 2z = 0\}$.

(l) Let V be an inner product space and T be a normal operator on V. Then prove that $\|T(x)\| = \|T^*(x)\| \ \forall\, x \in V$, where T^* is the adjoint of T.

(m) Find the dual basis β^* for V^* for the given basis $\beta = \{(1, 0, 1), (1, 2, 1), (0, 0, 1)\}$ of the real vector space $V = \mathbb{R}^3$.

(n) Let $\{v_1, v_2, ..., v_k\}$ be an orthogonal set in an inner product space V, and let $a_1, a_2, ..., a_k$ be scalars. Then prove that $\left\|\sum_{i=1}^{k} a_i v_i\right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2$.

(o) If V is a finite dimensional vector space over a field F, then prove that for any $v \neq 0$ in V there exists $g \in V^*$ such that $g(v) \neq 0$.

2. Answer *any four* questions: 5×4=20

(a) (i) For any prime number p, show that $x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1$ is irreducible over $\mathbb{Q}$.

(ii) $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} | a, b, \in \mathbb{Z}\}$. Show that 3, 1+ $2\sqrt{-5}$ are co-prime. 3+2

(b) (i) Show that the ring $R = \left\{\frac{m}{n} : m, n \in \mathbb{Z}, n \text{ is odd}\right\}$ is a PID.

(ii) Show that a subring of a PID need not be a PID. 3+2

(c) (i) Prove that $\langle X^2 + 1\rangle$ is a maximal ideal in $\mathbb{R}[X]$.

(ii) Show that $\frac{\mathbb{Q}[X]}{I}$, where $I = \langle x^2 - 5x + 6\rangle$ is not a field. 3+2

(d) (i) Let V be a finite dimensional vector space over the field F. Then prove that each basis for V^* is the dual of some basis for V.

(ii) Let V be a finite dimensional vector space over the field F. If L is linear functional on V^*, then prove that there exists a unique $v \in V$ such that $L(f) = f(v)$ for all $f \in V^*$. 3+2

(e) Let $\{e_k\}_{k=1}^{n}$ be an orthonormal sequence in an inner product space X. Then prove that for every $x \in X, \sum_{k=1}^{n} |\langle x, e_k\rangle|^2 \leq \|x\|^2$. 5

(f) Let V be an inner product space and let T be a linear operator on V. Then prove that T is an orthogonal projection if T has an adjoint T^* and $T^2 = T = T^*$. 5

3. Answer *any two* questions: 10×2=20

(a) (i) Prove that the polynomial f over real field $\mathbb{R}$ is a unit in $\mathbb{R}[X]$ iff f is a non-constant polynomial.

(ii) For any positive integer n, prove that $\mathbb{Z}_n[X]$ is an integral domain iff n is a prime number. 5+5

(b) (i) Let R be a PID. Then prove that any non-zero proper ideal of ring R can be expressed as finite product of maximal ideals of R.

(ii) Determine all the units of the ring $\mathbb{Z}[i]$ of Gaussian integers. 5+5

(c) (i) Let T be a linear operator on a vector space over a field F. Show that T need not be diagonalisable if the characteristic polynomial of T splits over F.

(ii) Apply Gram-Schmidt process to the given subset S of the inner product space $\mathbb{C}^3(\mathbb{C})$ to obtain an orthogonal basis for span (S) and then normalize the vectors in this basis to obtain an orthonormal basis β. Given, $S = \{(1, i, 0), (1 - i, 2, 4i)\}$. 5+5

(d) (i) Let T be a normal operator on an inner product space V over F. Then show that $T - CI$ is normal $\forall\, C \in F$.

(ii) Let T be a linear operator on a complex inner product space V with an adjoint T^*. Then prove that T is self-adjoint iff $\langle T(x), x\rangle$ is real for all $x \in V$. 5+5
