B.A./B.Sc. 6th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH6DSE41

(Bio Mathematics)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) What is the carrying capacity in logistic growth model?

(b) What is the Allee effect? Explain with an example.

(c) Discuss Gompertz growth model.

(d) Write a short note on Bacterial growth in a Chemostat.

(e) Determine when the steady state of the following equation is stable:

$$x_{n+1} = \frac{1}{2 + x_n}, n = 0, 1, 2, \ldots$$

(f) Find equilibrium solution of difference equation $x_{t+1} = rx_t(1 - x_t)$.

(g) Explain a continuous age-structured model.

(h) Discuss a simple discrete prey predator model.

(i) Explain the self-crowding effect in logistic growth model.

(j) What is intra-species competition? Explain with an example.

(k) Define a SIR model with generalized assumptions.

(l) What is the jury stability condition? Explain with an example.

(m) Write down the assumptions of density dependent growth models with harvesting.

(n) Discuss limit cycles with an example in the context of biological scenario.

(o) What are the assumptions of the Nicholson Bailey model?

2. Answer *any four* questions: 5×4=20

(a) Find the fixed points of the following model and carry out a linear stability analysis at (0, 0).

$$\frac{dN_1}{dt} = r_1N_1\left[1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right]$$
$$\frac{dN_2}{dt} = r_2N_2\left[1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_2}\right]$$

where $r_1, K_1, r_2, K_2, b_{12}$ and b_{21} are all positive constants and have their usual meanings.

(b) Obtain the condition under which the following model will have a positive interior equilibrium point.

$$\frac{dx}{dt} = x(1 - x - py)$$
$$\frac{dy}{dt} = y(1 - y - qx), \text{where } p, q > 0$$

(c) Consider the following non-linear difference equation for population growth:

$$x_{n+1} = \frac{kx_n}{b + x_n}, b, k > 0$$

Does equation have a steady state? If so, is that steady state stable? 3+2

(d) Solve the following initial value problem by the method of characteristics:

$$u_t + vu_x = 0, t \in (0, \infty), x \in (-\infty, \infty), u(0, x) = \phi(x), x \in (-\infty, \infty)$$

(e) Discuss the phase plane analysis of a two dimensional system $\dot{x} = ax + by, \dot{y} = cx + dy$ when the eigenvalues are complex conjugate, where *a*, *b*, *c* and *d* are real.

(f) What is diffusion in mathematical model? Give an example of two species model with diffusion.

3. Answer *any two* questions: 10×2=20

(a) State and prove the Routh-Hurwitz criteria for a second order polynomial equation. Hence, discuss the nature of the roots of characteristic equation of the following differential equation: (2+4)+4

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 3x = 0$$

(b) Consider the following system

$$\frac{dx}{dt} = x\left(1 - \frac{x}{k}\right) - dxy$$
$$\frac{dy}{dt} = exy - py$$

where *k*, *d*, *e* and *p* are all positive constants.

(i) Find corresponding steady states and Jacobean matrix around any fixed point.

(ii) Discuss the stability of interior steady state only. (3+2)+5

(c) Perform a local phase plane analysis of the following SIRS epidemic model:

$$\frac{dS}{dt} = -\frac{\beta}{N}SI - v(N - S - I),$$
$$\frac{dI}{dt} = \frac{\beta}{N}SI - \gamma I$$

where the parameters have their usual meanings. Find the equilibrium points and determine conditions for their local asymptotic stability. Consider two cases, $R_0 > 1$ and $R_0 \leq 1$, R_0 is the basic reproduction numbers. 4+6

(d) (i) Reduce a two species diffusion model into a linearized system around any specially uniform steady state.

(ii) Obtain the conditions for diffusive instability. 5+5

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B.A./B.Sc. 6th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH6DSE42

(Differential Geometry)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Define a space curve. Give an example of it.

(b) When is a curve in $\mathbb{R}^n$ said to be unit speed? Give an example of a unit speed curve.

(c) Define arc length of a plane curve. Find the arc length of the curve $\gamma(t) = (e^t \cos t, e^t \sin t)$ at $\gamma(0) = (1,0)$.

(d) Deduce the curvature of the curve $\gamma(t) = (\cos^3 t, \sin^3 t)$.

(e) Define signed curvature of a plane curve.

(f) Define a surface immersed in $\mathbb{E}^3$.

(g) Show that the surface of a sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a smooth surface.

(h) Deduce the first fundamental form of a plane.

(i) When is a curve on a surface said to be geodesic? Prove that any geodesic has constant speed.

(j) State second fundamental form of a surface $\sigma(u,v)$.

(k) When is a point on a surface said to be umbilic?

(l) State Meusnier's theorem.

(m) Give an example of a surface of positive curvature.

(n) Define mean curvature of a surface.

(o) Give an example of a surface whose Gaussian curvature and mean curvature are different.

2. Answer *any four* questions: 5×4=20

(a) If γ is a unit speed curve of $\mathbb{R}^3$ with constant curvature and zero torsion, then prove that γ is a part of a circle.

(b) Deduce the torsion of a curve $\gamma(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$.

(c) Deduce the first fundamental form of a surface $\sigma(u,v) = (u, v, u^2 + v^2)$.

(d) Prove that any tangent developable surface is isometric to a plane.

(e) Prove that a curve on a surface is a geodesic if and only if its geodesic curvature is zero everywhere.

(f) Prove that the Gaussian curvature of a ruled surface is negative or zero.

3. Answer *any two* questions: 10×2=20

(a) Let $\gamma(t)$ be a unit speed curve with $k(t) > 0$ and $\tau(t) \neq 0$ for all t. Show that γ lies on the surface of a sphere of radius r if and only if

$$\frac{\tau}{k} = \frac{d}{ds}\left(\frac{\dot{k}}{\tau k^2}\right),$$

where $r^2 = \rho^2 + (\dot{\rho}\sigma)^2, \rho = \frac{1}{k}, \sigma = \frac{1}{\tau}$ and dot $(\cdot)$ denotes the differentiation. 5+5

(b) Obtain a necessary and sufficient condition for a space curve to be a helix. 5+5

(c) State and prove Euler's theorem on a surface. 2+8

(d) Deduce the Gaussian curvature of the helicoid $\sigma(u, v) = (v \cos u, v \sin u, \lambda u)$ and catenoid $\sigma(u, v) = (\cos hu \cos v, \cos hu \sin v, u)$. 5+5

B.A./B.Sc. 6th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH6DSE43

(Mechanics II)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20 3

(a) Obtain the Lagrangian of a simple pendulum.

(b) Define a conservative field of force. Give an example. 1+1

(c) Find the equation of free surface when a fluid is in equilibrium under the action of gravity only.

(d) Show that the distance between two points remains invariant under Galilean transformation.

(e) Define Holonomic and Non-Holonomic constraints.

(f) State Archimedes principle for a floating body.

(g) Write down the necessary and sufficient condition for equilibrium of a fluid under the action of external forces.

(h) If the forces per unit mass at (x, y, z) parallel to the axes are $y(a-z), x(a-z), xy$, then examine whether the force field is in equilibrium or not.

(i) If a parallelogram be immersed in any manner in a homogeneous liquid, then prove that the sum of the pressures at the extremities of each diagonal is the same.

(j) What is an adiabatic change of state?

(k) Explain briefly the term "Convective Equilibrium".

(l) Find the work done in compressing a gas from volume V to volume U isothermally.

(m) Write down the stress matrix at a point in an ideal fluid with proper explanation of symbol.

(n) What is an isothermal process? Give example. 1+1

(o) Give the interpretation of D'Alembert's principle.

2. Answer *any four* questions: 5×4=20

(a) ABC is a triangular lamina with the side AB in the surface of a heavy homogeneous liquid. A point D is taken in AC, such that the thrusts on the areas ABD and DBC are equal. Find the ratio AD : AC.

(b) Deduce the relation, $\frac{T}{T_0} = 1 - \frac{\gamma-1}{\gamma}\frac{z}{H}$, assuming gravity to be constant.

(c) A particle of mass m moving in a central force field under inverse square law. Find its P.E. and K.E. Also obtain Lagrange's equation of motion. 1+1+3

(d) A small uniform circular tube, whose plane is vertical contains equal quantities of fluids whose densities are ρ and $\sigma(\rho > \sigma)$, and do not mix. If they together fill half of the tube, show that the radius passing through the common surface makes with the vertical an angle $\tan^{-1}\left(\frac{\rho-\sigma}{\rho+\sigma}\right)$.

(e) Obtain differential equation for curves of equipressure and equidensity.

(f) A hollow weightless hemisphere with a plane base is filled with water and hung by means of a string, one end of which is attached to a point of the rim on its base. Find the inclination to the horizontal of the resultant thrust on its curved surface.

3. Answer *any two* questions: 10×2=20

(a) (i) A given volume V of liquid is acted upon by forces $-\frac{\mu x}{a^2}, -\frac{\mu y}{b^2}, -\frac{\mu z}{c^2}$ ($\mu > 0$, a constant). Find the equation of the free surface.

(ii) A semi-circular lamina of radius a is immersed in a liquid with diameter in the surface. Find the depth of the centre of pressure. 5+5

(b) The Lagrangian L for the motion of a particle of unit mass is

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V + A\dot{x} + B\dot{y} + C\dot{z}$$

where each of V, A, B, C is a given function of (x, y, z). Show that the equations of motion can be written in the form

$$\ddot{\vec{r}} = -\vec{\nabla}V + \dot{\vec{r}} \times curl\vec{s}$$

where $\vec{r} = (x, y, z), \vec{s} = (A, B, C)$.

(c) (i) Define Galilean transformation.

(ii) Show that acceleration remains invariant under Galilean transformation.

(iii) The stress tensor at a point continuum is given by,

$$(\tau_{ij}) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Determine the principal stresses and the corresponding principal directions. 2+2+6

(d) (i) For a scleronomic dynamical system, show that the kinetic energy is a homogeneous quadratic function of generalized velocities.

(ii) A particle of mass m is projected in space with velocity v_0 at an angle α to the horizontal. Write the Lagrangian for the motion of the projectile and the equation of motion of the system. 6+(2+2)

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