B.A./B.Sc. 5th Semester (Honours) Examination, 2022 (CBCS)

Subject : Mathematics

Paper : BMH5CCXI

(Partial Differential Equation and Application)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

1. Answer *any ten* questions: 2×10=20

(a) Form a partial differential equation by eliminating the arbitrary functions f and g from $z = f(xe^y) + g(y^2 \cos y)$.

(b) Formulate a partial differential equation by eliminating arbitrary constants a and b from the equation $(x+a)^2 + (y+b)^2 = 1 - z^2$. Also write its order and degree.

(c) Show that $u(x,t) = e^{-\lambda^2\alpha^2.t}.(\cos\lambda x - \sin\lambda x)$ is a solution to the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$.

(d) Write down the Lagrange's linear PDE. What is the nature of this PDE?

(e) Find the general integral of $xzp + yzq = xy$.

(f) Write down one-dimensional wave equation and indicate its type. 1+1

(g) Obtain the characteristics curve of the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$.

(h) Determine the region where the partial differential equation $\frac{\partial^2 z}{\partial x^2} - (1+y)^2\frac{\partial^2 z}{\partial y^2} = 0$ is hyperbolic or parabolic. 1+1

(i) Determine whether the given partial differential equation is linear, semi linear, quasi linear or non-linear: 1+1

(I) $xp + yq = x^2 + y^2$

(II) $zp + q = 3$

(j) Find the solution of the PDE $\frac{\partial^2 z}{\partial y^2} = 36\frac{\partial^2 z}{\partial x^2}$ at the point (2, 1), subject to initial conditions at $z(x,0) = 5x$ and $\frac{\partial z}{\partial x}(x,0) = 0$.

(k) Show that if f and g are arbitrary functions of their respective arguments, then $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$, where $\alpha^2 = 1 - \frac{v^2}{c^2}$.

(l) Prove that the partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u$ can be transformed to $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ by using the transformation $v = e^{-t}.u$.

(m) Find the partial differential equation for the family of planes if the intercepts made by the arbitrary number of the family on the co-ordinate axes are $a, b, 1$ where $a \neq 0, b \neq 0$.

(n) State the Cauchy–Knowalevsky theorem.

(o) Solve the equation $(yz + z^2)dx - zxdy + xydz = 0$.

2. Answer *any four questions*: 5×4=20

(a) Find the integral surface of the linear PDE $xp + yq = z$ which contains the circle defined $x^2 + y^2 + z^2 = 4$ and $x + y + z = 2$, where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$.

(b) A homogeneous string is stretched and its ends are at $x = 0$ and $x = l$. Motion is started by displacing the string into the form $f(x) = z_0 \sin\left(\frac{\pi x}{l}\right)$, from which it is released at time $t = 0$. Find the displacement at any point x and time t.

(c) Solve the equation by the method of separation of variables $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial t} + u$ with $u(x, 0) = 5 \cdot e^{-2x}$.

(d) Solve the Cauchy problem by the method of characteristics $pz + q = 1$ with initial data $y = x, z = \frac{x}{2}$. Indicate the region where the solution is valid. 4+1

(e) Verify that the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ is reduced to the form $\frac{\partial^2 z}{\partial \alpha \partial \beta} = 0$ by the transformation $\alpha = \frac{1}{2}(x + y); \beta = \frac{1}{2}(x - y)$.

(f) Solve: $\frac{\partial^2 z}{\partial x \partial y} = e^{-x} \cos y$. Given that $z = 0$ when $x = 0$ and $\frac{\partial z}{\partial x} = 0$ when $y = 0$.

3. Answer *any two* questions: 10×2=20

(a) (i) Solve the equation by the method of separation of variables:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the conditions $u(0, y) = u(l, y) = u(x, 0) = 0, u(x, a) = \sin\frac{n\pi x}{l}$.

(ii) Equation of any cone with vertex (1, 2, 3) is of the form $f\left(\frac{x-1}{z-3}, \frac{y-2}{z-3}\right) = 0$. Find the partial differential equation represented by the above cone. 6+4

(b) (i) Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it.

(ii) Solve the Cauchy problem $2p + 3q = 1$ subject to $C: x_0 = t, y_0 = t, z_0 = t + 1$. 6+4

(c) (i) Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ subject to the boundary conditions $u(0,t) = 0$, $u(a,t) = 0$, for all t and the initial condition $u(x,0) = f(x), 0 < x < a$.

(ii) Find the general solution of the PDE $2y(z-3)\frac{\partial z}{\partial x} + (2x - z)\frac{\partial z}{\partial x} = y(2x-3)$. Hence find a solution which passes through the circle $z = 0$, $x^2 + y^2 = 2x$. 4+6

(d) (i) Define Cauchy problem for second order partial differential equation with example.

(ii) Find the general solution of the PDE $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = z$.

(iii) Verify that the equation $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0$ reduced to the form $\frac{\partial z}{\partial \theta} = 0$ by the transformation $x = r\cos\theta$, $y = r\sin\theta$. 2+4+4
