B.A/B.Sc 6th Semester (Honours) Examination, 2020 (CBCS)
Subject: Mathematics
Course: BMH6CC14
(Ring Theory and Linear Algebra-II)

Time: 3 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: 6×5=30

(a) Give an example of an ideal in $\mathbb{Z}[x]$ which is not a principal ideal. 5

(b) Prove that the polynomial $f(x) = 1 + x + x^2$ is irreducible over $\mathbb{Z}_2$. 5

(c) Prove that a commutative ring R with unity is a field when $R[x]$ is a principal ideal domain. 5

(d) Prove that a nonzero proper ideal of a principal ideal domain R is prime if and only if it is maximal. 5

(e) If V is a finite dimensional vector space, then show that there exists a canonical isomorphism from V onto V^{**}. 5

(f) Prove that any orthogonal set of non-null vectors in an inner product space is linearly independent. 5

(g) Apply Gram Schmidt process to the set of vectors $\{(1,0,1), (1,0,-1), (1,3,4)\}$ to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product. 5

(h) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation, defined by $T(x, y, z) = (2x + y - 2z,\ 2x + 3y - 4z,\ x + y - z)$. Find the eigen values of T. 5

2. Answer any three questions: 3×10=30

(a) (i) Determine all the units of the ring $\mathbb{Z}[i]$ of Gaussian integers.

(ii) If α and β be vectors in an inner product space, then show that

$\|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2$. 5+5

(b) (i) Show that 3 is irreducible but not prime in $\mathbb{Z}[i\sqrt{5}] = \{a + ib\sqrt{5} : a, b \in \mathbb{Z}\}$.

(ii) Verify Cayley-Hamilton Theorem for the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$. 5+5

(c) (i) Let A be an $n \times n$ symmetric matrix over $\mathbb{R}$ and suppose that $\mathbb{R}^n$ is equipped with the standard inner product. If $< u, Au > = < u, u >, \forall\, u \in \mathbb{R}^n$, then prove that $A = I_n$.

(ii) Prove that $F[x]$ is an Euclidian domain for a field F. 5+5

(d) (i) Suppose W_1 and W_2 are two subspaces of a finite dimensional vector space V. Prove that $(W_1 + W_2)^0 = {W_1}^0 \cap {W_2}^0$, where W^0 is annihilator of W.

(ii) Let $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$ .Then prove that $\mathbb{Z}[\sqrt{3}]$ is FD. 5+5

(e) (i) Suppose $V = \{a + bt : a, b \in \mathbb{R}\}$, the vector space of real polynomials of degree ≤ 1. Let $\theta_1\big(f(t)\big) = \int_0^1 f(t)\,dt$ and $\theta_2\big(f(t)\big) = \int_0^2 f(t)\,dt$. Show that $S = \{\theta_1, \theta_2\}$ is a basis of V^*. Find a basis of V for which S is the dual basis.

(ii) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator whose matrix representation with respect to the standard ordered basis $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$ is $\begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix}$. Find the minimal polynomial of T.

6+4